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Brane webs and 1/4-BPS geometries

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Abstract: We discuss brane webs preserving eight supercharges and derive geometries produced by them. Consistency conditions of supergravity are shown to impose certain requirements on the locations of the sources, and these restrictions are found to be in a perfect agreement with results of the probe analysis. In particular, solutions of IIB SUGRA describing (p, q) stings are inconsistent, unless the web consists of straight line segments whose orientation is correlated with charges of the string. The geometries produced by membranes and D3 branes are only consistent if brane profiles are holomorphic. Using perturbation theory, we show that a unique gravity solution exists for any allowed distribution of sources. We also revisit 1/4-BPS geometries with $AdS_p \times S^q$ asymptotics and derive the boundary conditions leading to regular geometries. All degenerate limits of regular solutions are shown to agree with expectations from the brane probe analysis.

Keywords: Brane Dynamics in Gauge Theories, D-branes, AdS-CFT Correspondence.

Contents

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1. Introduction and summary

Brane intersections have been instrumental in improving our understanding of black holes [1] and in constructing string realization of interesting field-theoretic phenomena [2]. While the vast majority of work on intersecting branes has been devoted to orthogonal intersections, studies of branes intersecting at angles, initiated in [3],[1] led to significant progress in classifying supersymmetric objects in string theory and to applications to model building [5]. The situation becomes especially interesting if intersecting branes have the same dimensionality of the worldvolume: in this case one can get complicated systems known as "brane webs" [6, 7]. In particular, string webs have emerged in the descriptions of the dyonic black holes [8], so by finding a geometry produced by an individual web, one can provide a gravitational representation of a microscopic state contributing to the black hole entropy. String webs also provide a nice geometric interpretation of various field-theoretic phenomena [9]. The webs of higher-dimensional branes are even more interesting: unlike the (p, q) strings, which follow straight lines, these objects can have very complicated shapes. Brane webs occupy a very special niche among curved branes since no gauge field is turned on along their worldvolume (usually such gauge field acts as a cause of a nontrivial shape, see [10] for a discussion of a prototypical example), and they clearly deserve better understanding.

Most of the work on curved D-branes has been performed using an open string picture for the branes [11]. In particular, this description has been used to determine the shapes of the branes and to analyze the supersymmetries preserved by them. However, D-branes have also emerged as solitons in the closed string theory [12], which gives an alternative way of studying their dynamics. Moreover, since the open– and closed-string pictures describe the same object in different corners of the parameter space [13], a comparison of brane dynamics derived from these complementary descriptions should provide a test of an open/closed string duality proposed in [13]. In the last decade many such checks have been performed, but mainly they were done in the decoupling limit, where open string

[1]See also [4] for the discussion of intersections in M theory.

description reduces to field theory [14 − 16]. It would be very nice to extend this agreement to the full-fledged duality between open and closed strings.

Unfortunately, in the presence of Ramond-Ramond fluxes, genuine string computations are very hard, if not impossible, so often one has to rely on low-energy effective actions: supergravity for closed strings, and DBI action for the open ones. For orthogonal brane intersections preserving eight supercharges, a perfect agreement between SUGRA and DBI analyses was found in [17], and one of the goals of this paper is to extend the results of that work to the brane webs. To perform a comparison, we will construct the geometries produced by various webs, and these metrics might have numerous applications even apart from testing DBI/SUGRA duality. Once the agreement between two descriptions is demonstrated in the asymptotically-flat case, one is naturally led to a correspondence between various quantities in the near-horizon limit.

In this article we will study the webs constructed from branes with the same dimensionality, and three such systems admit regular near-horizon limits: the webs of D3, M2 and M5 branes. Even for a single stack of branes, a near-horizon limit of the geometry is not geodesically-complete, and, to recover the entire $AdS_p \times S^q$, some continuation of the metric is required [16]. In the case of 1/4-BPS brane webs, an analogous extension leads to the geometries whose local structure has been determined in [18, 19]. Such continuation corresponds to formulating field theory on $R \times S^{p-2}$ rather than on $R^{1,p-2}$, while on the gravity side it removes infinite throats and replaces them by smooth "bubbles". For example, the vacuum of the field theory on $R \times S^{p-2}$ corresponds to the $AdS_p \times S^q$ with global coordinates on AdS space, and 1/2-BPS states in this theory correspond to the smooth geometries discovered in [20]. Similarly, 1/4-BPS states in field theory correspond to some of the solutions constructed in [18, 19], but, according to the rules of AdS/CFT correspondence, only regular geometries (and their degenerate limits) are allowed. This implies that the local analysis of [18, 19] should be supplemented by some boundary conditions, and we will derive them in this paper. As in the 1/2-BPS case [20], the boundary conditions can be formulated in terms of droplets in some Kahler space, but, unlike their 1/2-BPS cousins, the droplets corresponding to 1/4-BPS solutions cannot have arbitrary shapes. We will derive the conditions which should be obeyed by the boundaries of droplets, and these restrictions will be shown to agree with expectations from the analysis of brane probes. This will serve as one of the checks of the open/closed duality in the near-horizon region.

This article has the following organization. In section 2 we review the description of brane webs in terms of open strings. In particular, we classify the nontrivial 1/4-BPS webs in IIB string and M theories and show that they either form planar networks built from straight lines, or follow holomorphic profiles. While most of the results presented in that section are well-known, it is useful to write them in the uniform notation for later comparison with gravity analysis.

In section 3 we derive the geometries produced by the webs of (p, q)-strings, and we demonstrate that, for consistency of SUGRA, such webs must be built from straight line segments, and orientation of the segments must be correlated with charges p and q. Although this conclusion comes from supergravity (and it does not rely on any information about branes), it agrees perfectly with outcome of the probe (or open-string) analysis,

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and this agreement can be viewed as a nontrivial check of the open/closed string duality. Section 4 extends such agreement to the webs of M2 branes, where situation is even more interesting: worldvolume analysis suggests that the membranes must follow holomorphic profiles, and we confirm this result by an independent computation in supergravity. In section 5 we use various dualities to construct geometries produced by other brane webs, and we compare with earlier results of [17]. Again, a perfect agreement between probe and SUGRA descriptions is found.

Notice that the metrics discussed in section 4 have been written before [21], but they have never been derived from the first principles. The authors of [21] proposed an ansatz which contained a Kahler space fibered over $\mathbf{R}^6$, enforced the projectors which were exported from the probe analysis, and checked that all SUGRA equations were satisfied. While this approach yielded a solution of eleven-dimensional supergravity, it was not clear whether there were any generalizations, moreover, since the open-string projectors were introduced by hand, the construction of [21] could not provide an independent check of the open/closed string duality. In contrast to the approach of [21], our derivation is based only on supergravity, so an agreement with probe analysis appears as a nontrivial agreement.

The authors of [21] also argued that a solution produced by a curved M2 brane did not exist. In particular, perturbative arguments were used to conclude that gravity solution breaks down *everywhere* in space. However, this statement is somewhat counter-intuitive, since, starting with probe approximation, one should be able to turn on gravity without creating problems sufficiently far away from the branes. In section 4.2 we show that the divergences encountered in [21] originate from the perturbation theory in charges which was introduced in that paper, but, performing a more natural multipole expansion instead, one arrives at a well-defined perturbation series which converges away from the sources. We view this fact as a strong evidence pointing to the existence of the supergravity solution, and we argue that the "improved" perturbation theory produces a unique geometry for any allowed distribution of membranes.

The second part of this paper is devoted to studying 1/4-BPS geometries with $AdS_p \times S^q$ asymptotics. While such solutions can be constructed by taking decoupling limits of the brane webs discussed in sections 4, 5, the resulting space would not be geodesically complete. This situation has also been encountered for a single stack of D3 branes, and in that case the space can be continued to produce a global $AdS_5 \times S^5$. As a result of such continuation, one finds a string dual of a field theory on a sphere [16]. Unfortunately, it is not clear how to perform a similar continuation for the near-horizon geometry of a brane web, but, fortunately, the local structure of the desired solutions have been derived from the first principles [18, 22, 19]. In section 6 we analyze the global properties of the geometries constructed in [22, 19] and determine the restrictions imposed by regularity.

The solutions of [22, 19] have $\mathrm{SO}(4) \times \mathrm{U}(1) \times \mathrm{U}(1)_t$ symmetry, and the metric contains a coordinate y which goes to zero when either S^3 or $\mathrm{U}(1)$-direction collapses to zero size. Then, requiring regularity of the solution at $y = 0$, one arrives at some boundary conditions on this hyperplane. This situation looks similar to the one encountered in the 1/2-BPS case [20], where regularity implied that one-dimensional Kahler space was divided into droplets by assigning one of two values ($Z = \pm\frac{1}{2}$) to a certain function. This analogy led the authors of [23] to propose a similar picture for the 1/4-BPS droplets: it was argued that, by introducing droplets with $Z = \pm\frac{1}{2}$ in 2D Kahler space, one produces a regular geometry. It turns out that the 1/4-BPS case is more subtle. First, in addition to requiring certain value for function Z, one needs to impose a restriction on Kahler potential in $Z = -\frac{1}{2}$ regions (see equation (6.109)):

$$y = 0: \quad \begin{array}{l} Z = -\frac{1}{2},\ \partial_a \bar{\partial}_b K(z, \bar{z}, y = 0) = 0, \\ \\ Z = +\frac{1}{2}. \end{array} \tag{1.1}$$

The second subtlety is associated with the shapes of the droplets: while they could be arbitrary in the 1/2-BPS case [20], now they are restricted by the regularity conditions. To be more precise, describing the wall between droplets by an equation $v(z_a, \bar{z}_a) = 0$, one finds a restriction (6.110) on a real function v:

$$\partial_a \bar{\partial}_b v + \lambda \partial_a v \bar{\partial}_b v = g \partial_a w \bar{\partial}_b \bar{w} + O(v), \quad \det(\partial_a v \partial_b w)|_{v=0} \neq 0. \tag{1.2}$$

Function w defined here must be holomorphic. The conditions (1.1) and (1.2) are derived in section 6.5.

While the relation (1.2) was derived from SUGRA analysis, it is also crucial for ensuring an agreement with probe calculations. As was shown in [24], the profiles of supersymmetric D3 branes in $AdS_5 \times S^5$ are described by holomorphic surfaces. In section 6.4 we extend this result to branes on an arbitrary 1/4-BPS background,[2] but, starting from an arbitrary droplet and contracting it, one can potentially arrive at the sources which do not have holomorphic profiles. In section 6.5 we show that it is the restriction (1.2) which prevents this from happening, and that regular droplets can collapse only to holomorphic cycles (which can be arbitrary). Moreover, in sections 6.3), (6.4 we also show that holomorphicity of the brane embeddings can be independently derived both from probe analysis and from consistency of SUGRA, so, in some sense, a lack of the restriction (1.2) would have implied an internal inconsistency of supergravity.

In section 6.6 we demonstrate that, for the fixed asymptotic behavior, any distribution of droplets with boundary conditions (1.1) leads to the unique geometry, and the restriction on the Kahler potential appearing in (1.1) is instrumental in ensuring the uniqueness. Of course, to avoid extra sources on the domain walls, the restrictions (1.2) should also be imposed.

In section 6.7 we discuss the topology of the 1/4-BPS solutions: we construct two types of non-contractible five-manifolds, and find very simple expressions for the flux of F_5 through these cycles. This is very similar to the picture encountered in the 1/2-BPS case [20]: rather than having sources, the fluxes are supported by non-trivial topology.

Section 7 is devoted to the discussion of 1/4-BPS geometries in M theory, which are obtained by an analytic continuation of the metrics found in [18]. Since the resulting geometries share many qualitative properties with their ten-dimensional cousins, section 7 is rather brief: we only underline the differences. While we were not able to construct new

[2]We also provide a simple geometric interpretation of the radial coordinate introduced in [24].

explicit solutions in ten or eleven dimensions, sections 6.2 and 7.2 discuss several examples which are obtained by embedding some old solutions into new ansatze. In particular, we embed all geometries constructed in [20]. While doing this embedding, one notices that there are striking similarities between ten– and eleven-dimensional solutions, which are not obvious at first sight. In section 8 we rewrite type IIB solutions (both in 1/2– and 1/4-BPS case) in a way which makes an analogy with M theory more transparent.

2. Webs in the probe approximation

We begin with recalling some well-known facts about supersymmetric brane webs in IIB string theory. Supersymmetry transformations in this theory are parameterized by two Majorana-Weyl spinors which have the same chirality, and it is convenient to combine them into a 32-component real object ϵ which satisfies a chirality projection:

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad \mathbf{1}_2 \otimes \Gamma_{11}\epsilon = -\epsilon : \qquad \Gamma_{11}\epsilon_{1,2} = -\epsilon_{1,2}. \tag{2.1}$$

Ten-dimensional flat space preserves 32 supersymmetries corresponding to arbitrary constant values of ϵ_1 and ϵ_2 (modulo the chiral projection). By adding a brane to $R^{9,1}$ one breaks half of the supersymmetries and the appropriate projections are [25] (see also [26] for a review):

$$\begin{aligned} &\text{F1}: && \Gamma = \sigma_3 \otimes \Gamma_{(2)}, && \Gamma\epsilon = \epsilon, \\ &\text{NS5}: && \Gamma = \sigma_3 \otimes \Gamma_{(6)}, && \Gamma\epsilon = \epsilon, \\ &\text{D}(2p-1): && \Gamma = i\sigma_3^p \sigma_2 \otimes \Gamma_{(2p)}, && \Gamma\epsilon = \epsilon. \end{aligned} \tag{2.2}$$

Here $\Gamma_{(2p)}$ is a product of gamma matrices with indices pointing along the worldvolume of the brane.

Each of the branes appearing in (2.2) preserves 16 real supercharges and there are two other interesting objects which have the same amount of SUSY — a plane wave and a KK monopole:

$$\begin{aligned} &\text{P}: && \Gamma = \mathbf{1}_2 \otimes \Gamma_{(2)}, && \Gamma\epsilon = \epsilon, \\ &\text{KK}: && \Gamma = \mathbf{1}_2 \otimes \Gamma_{(6)}, && \Gamma\epsilon = \epsilon. \end{aligned} \tag{2.3}$$

These configurations have a pure geometric nature and they do not involve fluxes.

Once the building blocks preserving half of the supersymmetries are specified, one can construct supersymmetric intersections by combining the ingredients with commuting projectors. In particular, it is interesting to look at the orthogonal intersections of branes which have the same number of worldvolume directions:

$$\begin{pmatrix} F1_1 \\ D1_2 \end{pmatrix}, \begin{pmatrix} D3_{123} \\ D3_{145} \end{pmatrix}, \begin{pmatrix} D5_{12345} \\ D5_{12367} \end{pmatrix}, \begin{pmatrix} D5_{12345} \\ D5_{16789} \end{pmatrix}, \begin{pmatrix} NS5_{12345} \\ D5_{12678} \end{pmatrix}, \begin{pmatrix} NS5_{12345} \\ D5_{12346} \end{pmatrix}, \tag{2.4}$$

All these configurations preserve eight real supercharges. It turns out that the same supercharges are preserved by more general configurations which are known as brane webs [6, 7]. Let us analyze this in more detail.

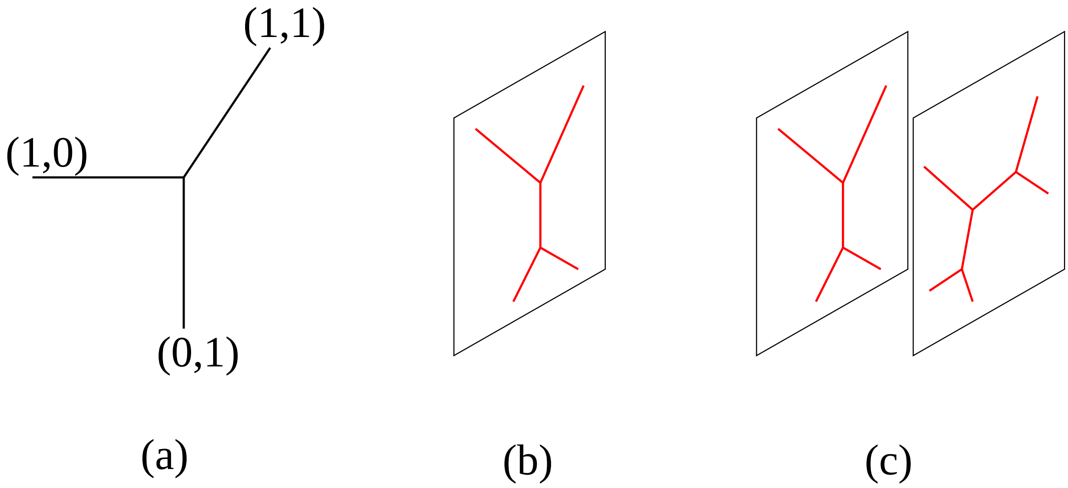

Figure 1: String webs: (a) an elementary junction, (b) a connected web, (c) a generic supersymmetric web.

String web. We begin with the first system in (2.4). Any spinor satisfying the $F1_1$ and $D1_2$ projectors also obeys the following relations:

$$\Gamma_{(p,q)}\epsilon = \epsilon,$$
$$\Gamma_{(p,q)} = \left[\frac{p}{\sqrt{p^2+\frac{q^2}{g_s^2}}}\sigma_3 + \frac{g_s^{-1}q}{\sqrt{p^2+\frac{q^2}{g_s^2}}}\sigma_1\right] \otimes \Gamma_0 \left[\frac{p}{\sqrt{p^2+\frac{q^2}{g_s^2}}}\Gamma_1 + \frac{g_s^{-1}q}{\sqrt{p^2+\frac{q^2}{g_s^2}}}\Gamma_2\right]. \quad (2.5)$$

The $\Gamma_{(p,q)}$ projector corresponds to a (p,q)-string which carries p units of string charge and q units of D1 charge[3] [6] (this can be read off from the first bracket in $\Gamma_{(p,q)}$). The second bracket in (2.5) indicates that the string stretches along the line

$$qx_1 - g_s p x_2 = \text{const}, \quad x_3, \ldots x_9 \text{ — fixed.} \quad (2.6)$$

The (p,q) strings which are mutually BPS can be joined to produce complicated string webs [6], which are constrained only by charge conservation at the junctions.

In particular, it is interesting to consider a junction with incoming fundamental string and D1 brane (see figure 1a) and compare it with a general picture for strings ending on branes [10]. By charge conservation, the outgoing leg should be a $(1,1)$ string which has a trajectory

$$x_1 = g_s x_2, \quad (2.7)$$

and, as expected, for small string coupling this is a small perturbation of a vertical D1 brane. The general analysis of [10] suggests that the resulting $(1,1)$ string can also be viewed as a D1 brane with some electric flux on its worldvolume and a nontrivial profile $x_1(x_2)$:

$$E = \partial_2 x_1, \quad \partial_2^2 x_1 = 0 \quad (2.8)$$

[3]For simplicity we consider the case of vanishing axion: $\tau \equiv \frac{i}{g_s} + a = \frac{i}{g_s}$.

Clearly the profile (2.7) solves the Laplace equation, then (2.8) gives an expression for the electric field. By analyzing the coupling between the electric field and the Kalb-Ramond tensor in the bulk, one can see that the proper string charge is also reproduced.[4] Thus we see that the string junction can be viewed as a simplest example of a bion.

Starting from any junction on the string web, one can draw a plane through the strings which pass through it, and define Cartesian coordinates (x_1, x_2) in this plane. Then the projection (2.5) implies that all legs of the supersymmetric web must be parallel to this plane, so any connected string web must be planar (see figure 1b). However it is also possible to have disconnected webs which can be constructed by combining elementary webs oriented in the same directions (see figure 1c). To construct the geometries produced by the (p, q) systems, it is convenient to begin with metric produced by an elementary block depicted in figure 1b, and then generalize it to the case of disconnected webs. This logic will be implemented in section 3.

D3-web. Let us now consider the D3-D3 system which appears in (2.4). The preserved supersymmetries satisfy two independent relations, and it is useful to combine them in the following way:

$$i\sigma_2 \otimes \Gamma_{0123}\epsilon = i\sigma_2 \otimes \Gamma_{0145}\epsilon = \epsilon : \quad \Gamma_{2345}\epsilon = -\epsilon. \tag{2.9}$$

The last projector allows us to write two more conditions:

$$i\sigma_2 \otimes \Gamma_{0124}\epsilon = -i\sigma_2 \otimes \Gamma_{0135}\epsilon, \quad i\sigma_2 \otimes \Gamma_{0125}\epsilon = i\sigma_2 \otimes \Gamma_{0134}\epsilon, \tag{2.10}$$

and the relations listed in (2.9), (2.10) can be summarized in a compact form:

$$i\sigma_2\Gamma_{01a\bar{b}}\epsilon = \frac{i}{2}\delta_{a\bar{b}}\epsilon, \quad z^1 = x_2 + ix_3, \quad z^2 = x_4 + ix_5. \tag{2.11}$$

As in the case of F1-D1 intersections, we find that this Killing spinor is preserved by a larger family of branes. To see this, we recall the kappa-symmetry projection associated with a D brane[5] [25, 27]:

$$\begin{aligned} \mathrm{D}(2p-1):\ \ \Gamma = i\sigma_3^p\sigma_2 \otimes \left[\mathcal{L}^{-1}\left(\prod_{m=0}^{2p-1}\frac{\partial X^{\mu_m}}{\partial\xi^m}\right)\Gamma_{\mu_0\dots\mu_{2p-1}}\right], \quad \Gamma\epsilon = \epsilon,\\ \mathcal{L} = \sqrt{\det(G_{\mu\nu}\partial_m X^\mu\partial_n X^\nu)}. \end{aligned} \tag{2.12}$$

In the case of D3 branes it is convenient to introduce complex coordinates both in spacetime and on the worldvolume:

$$Z^1 = X_2 + iX_3, \quad Z^2 = X_4 + iX_5, \quad w = \xi_2 + i\xi_3 \tag{2.13}$$

[4] A similar analysis for a string ending on D3 brane was performed in [17].

[5] We assumed that the branes have no fluxes on their worldvolume, but a more general case can be considered as well.

Then, assuming that Z^a are functions of w, $\bar{w}$ and imposing the static gauge in the remaining two directions ($X^0=\xi^0$, $X^1=\xi^1$), we find

$$\Gamma\epsilon = i\sigma_2\Gamma_{01}\frac{1}{\sqrt{\det(\partial_m Z^a\partial_n\bar{Z}^a)}}\left[(\partial Z^b\,\bar{\partial}\bar{Z}^{\bar{c}}-\bar{\partial}Z^b\,\partial\bar{Z}^{\bar{c}})\Gamma_{b\bar{c}}+\partial Z^b\,\bar{\partial}Z^c\Gamma_{bc}+\partial\bar{Z}^{\bar{b}}\,\bar{\partial}\bar{Z}^{\bar{c}}\Gamma_{\bar{b}\bar{c}}\right]\epsilon$$
$$\tag{2.14}$$
$$=\frac{2i}{\sqrt{\det(\partial_m Z^a\partial_n\bar{Z}^a)}}\left[\frac{1}{4}(\partial Z^b\,\bar{\partial}\bar{Z}^b-\bar{\partial}Z^b\,\partial\bar{Z}^b)+\partial Z^b\,\bar{\partial}Z^c\Gamma_{bc2\bar{2}}+\partial\bar{Z}^{\bar{b}}\,\bar{\partial}\bar{Z}^{\bar{c}}\Gamma_{\bar{b}\bar{c}2\bar{2}}\right]\epsilon.$$

To arrive at the second line we used the projector (2.11). The right-hand side of the last expression is proportional to ϵ if and only if

$$\partial Z^a\bar{\partial}Z^b=0 \tag{2.15}$$

for all values of a and b. This implies that Z^1 and Z^2 must be holomorphic functions,[6] then the relation (2.14) simplifies:

$$\Gamma\epsilon=\frac{1}{\sqrt{\frac{i}{2}\partial Z^a\bar{\partial}\bar{Z}^a}}\left[\frac{i}{2}\partial Z^b\,\bar{\partial}\bar{Z}^b\right]\epsilon=\epsilon \tag{2.16}$$

Thus we learned that a spinor satisfying the projection (2.11) is preserved by any D3 brane which follows a holomorphic profile in (Z^1,Z^2) directions. In particular, we can consider a straightforward generalization of (2.6):

$$Z_1-aZ_2=\text{const}, \tag{2.17}$$

which corresponds to a D3 brane with flat worldvolume, and one can form "D3-webs" by constructing the junctions of such objects. Of course, more general holomorphic profiles:

$$f(Z_1,Z_2)=0 \tag{2.18}$$

can also be considered, but we will refer to them as D3-webs as well.

Webs of D5-branes. The third configuration in (2.4) has eight supercharges which are also preserved by an arbitrary holomorphic web $x_4+ix_5=x_6+ix_7$. The simplest way to see this in to notice that two T dualities relate this configuration of five-branes to the D3-D3 system discussed above.

The spinors preserved by the fourth configuration in (2.4) obey two independent projections:

$$\sigma_1\otimes\Gamma_{012345}\epsilon=\epsilon,\quad \sigma_1\otimes\Gamma_{016789}\epsilon=\epsilon. \tag{2.19}$$

To analyze whether these projections can be satisfied by any nontrivial 1/4-BPS brane webs, it is convenient to introduce complex variables:

$$Z^1=x_2+ix_3,\quad Z^2=x_4+ix_5,\quad Z^3=x_6+ix_7,\quad Z^4=x_8+ix_9. \tag{2.20}$$

[6]Alternatively, one could have assumed that these functions are anti-holomorphic, but this would lead to $\Gamma\epsilon=-\epsilon$, i.e. such configurations preserve the wrong spinor.

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In terms of these coordinates the projectors become

$$\sigma_1\otimes\Gamma_{01}\Gamma_{Z_1\bar{Z}_1Z_2\bar{Z}_2}\epsilon=-\frac{1}{4}\epsilon,\quad \sigma_1\otimes\Gamma_{01}\Gamma_{Z_3\bar{Z}_3Z_4\bar{Z}_4}\epsilon=-\frac{1}{4}\epsilon. \tag{2.21}$$

Notice that, since we are looking for 1/4-BPS configurations, these relations must be satisfied by both $\epsilon_-=\frac{1}{2}(1-\sigma_1\otimes\Gamma_{01})\epsilon$ and $\epsilon_+=\epsilon-\epsilon_-$, so we can concentrate on ϵ_-. If one introduces a basis corresponding to each one of the complex coordinates:

$$\Gamma_Z|\downarrow\rangle=0,\quad \Gamma_Z|\uparrow\rangle=|\downarrow\rangle,\quad \Gamma_{\bar{Z}}|\uparrow\rangle=0,\quad \Gamma_{\bar{Z}}|\downarrow\rangle=|\uparrow\rangle, \tag{2.22}$$

then an arbitrary spinor ϵ_- satisfying projection (2.21) can be written as

$$\epsilon_-=e_1|\uparrow\uparrow\uparrow\uparrow\rangle+e_2|\uparrow\uparrow\downarrow\downarrow\rangle+e_3|\downarrow\downarrow\uparrow\uparrow\rangle+e_4|\downarrow\downarrow\downarrow\downarrow\rangle. \tag{2.23}$$

To preserve a quarter of supersymmetries, a brane should admit Killing spinors with independent coefficients e_i. Application of the projector (2.12) to the spinor ϵ_- leads to the relation

$$\Gamma\epsilon_-=\mathcal{L}^{-1}\left(\prod_{m=1}^{4}\frac{\partial X^{\mu_m}}{\partial\xi^m}\right)\Gamma_{\mu_1\dots\mu_4}\epsilon_-\equiv P[X^{\mu_1},X^{\mu_2},X^{\mu_3},X^{\mu_4}]\Gamma_{\mu_1\mu_2\mu_3\mu_4}\epsilon_-, \tag{2.24}$$

and the right-hand side of this expression should be equal to ϵ_-. Looking at coefficients in front of various components of the spinor and requiring them to match for arbitrary values of e_1, e_2, e_3, e_4, we find several relations:

$$\begin{aligned}
&P[Z_1,Z_2,Z_3,Z_4]=0,\quad P[Z_a,Z_b,Z_c,\bar{Z}_c]=0,\quad \sum_{a,b}P[Z_a,\bar{Z}_a,Z_b,\bar{Z}_b]=\frac{1}{4}\\
&P[Z_1,Z_2,\bar{Z}_3,\bar{Z}_4]=\sum_a P[Z_1,Z_2,\bar{Z}_a,Z_a]=\sum_a P[Z_3,Z_4,\bar{Z}_a,Z_a]=0\\
&\sum_{a,b}s_as_bP[Z_a,\bar{Z}_a,Z_b,\bar{Z}_b]=\frac{1}{4},\quad s_1=s_2=1,\quad s_3=s_4=-1
\end{aligned} \tag{2.25}$$

The first equation implies that (Z_1,Z_2,Z_3,Z_4) are functionally-dependent, i.e. we can choose a patch where $Z_4=f(Z_1,Z_2,Z_3)$. The second equation in (2.25) implies a functional dependence between (Z_1,Z_2,Z_3) and any one of the three functions $\bar{Z}_1$, $\bar{Z}_2$, $\bar{Z}_3$, in particular we find

$$Z_3=f_3(Z_1,Z_2,\bar{Z}_1)=\tilde{f}_3(Z_1,Z_2,\bar{Z}_2). \tag{2.26}$$

On the chosen patch we can also impose a static gauge by identifying $(Z_1,Z_2,\bar{Z}_1,\bar{Z}_2)$ with coordinates along the brane. Since these four coordinates are functionally independent, the last equation can only be satisfied if f_3 is a holomorphic function, then we find

$$Z_3=f_3(Z_1,Z_2),\quad Z_4=f_4(Z_1,Z_2). \tag{2.27}$$

The two non-homogeneous equations from (2.25) can be combined produce a simple relation:

$$0=\sum_{a=1}^{2}\sum_{b=3}^{4}P[Z_a,\bar{Z}_a,Z_b,\bar{Z}_b]=\sum_{a=1}^{2}\sum_{b=3}^{4}\left|\frac{\partial Z_b}{\partial Z_a}\right|^2P[Z_1,\bar{Z}_1,Z_2,\bar{Z}_2], \tag{2.28}$$

then, since $P[Z_1, \bar{Z}_1, Z_2, \bar{Z}_2] \neq 0$, we conclude that both Z_3 and Z_4 must be constants on our patch. This brings us back to the flat brane $D5_{12345}$ which appeared in (2.4). Thus we see that, unlike the first three systems in (2.4), the $1+1$-dimensional orthogonal D5-D5 intersection does not have any interesting generalization analogous to the string webs.

Since D5-D5 intersection and the fifth configuration in (2.4) can be obtained from the same M theory system:

$$\begin{pmatrix} D5_{12345} \\ D5_{16789} \end{pmatrix} \overset{R_M, T_1}{\longleftarrow} \begin{pmatrix} M5_{M2345} \\ M5_{M6789} \end{pmatrix} \overset{R_6, T_2}{\longrightarrow} \begin{pmatrix} NS5_{M2345} \\ D5_{M26789} \end{pmatrix}, \tag{2.29}$$

we conclude that NS5 and D5 branes intersecting along $2+1$-dimensional subspace do not have interesting generalizations.

(p,q)-fivebranes. The last configurations in (2.4) is very similar to the F1-D1 system: in this case an analog of the relation (2.5) can be written as:

$$\Gamma_{(p,q)}\epsilon = \epsilon,$$
$$\Gamma_{(p,q)} = \left[\frac{g_s^{-1}p}{\sqrt{q^2+\frac{p^2}{g_s^2}}}\sigma_3 + \frac{q}{\sqrt{q^2+\frac{p^2}{g_s^2}}}\sigma_1\right] \otimes \Gamma_0 \left[\frac{g_s^{-1}p}{\sqrt{q^2+\frac{p^2}{g_s^2}}}\Gamma_5 + \frac{q}{\sqrt{q^2+\frac{p^2}{g_s^2}}}\Gamma_6\right]\Gamma_{1234}$$

This projection corresponds to a (p,q) five-brane [7] stretching in the x_1, x_2, x_3, x_4 directions and along the line

$$px_6 - g_s q x_5 = \text{const.} \tag{2.30}$$

This completes our discussion of the 1/4-BPS brane webs appearing in IIB string theory and now we will make few comments about similar systems in eleven dimensions.

Brane webs in M theory. It is easy to classify the orthogonal intersections of M5 (M2) branes which preserve eight supercharges:

$$\begin{pmatrix} M5_{12345} \\ M5_{12367} \end{pmatrix}, \quad \begin{pmatrix} M5_{12345} \\ M5_{16789} \end{pmatrix}, \quad \begin{pmatrix} M2_{12} \\ M2_{34} \end{pmatrix}. \tag{2.31}$$

The first intersection is related by U-duality to the D3-D3 system which was discussed above, so it preserves the same supercharges as a web of the M5-branes with harmonic profiles:

$$x_4 + ix_5 = f(x_6 + ix_7), \quad x_8 = x_8^{(0)}, \ x_9 = x_9^{(0)}, \ x_{10} = x_{10}^{(0)}. \tag{2.32}$$

The second intersection in (2.31) is related to the D5-D5 system (see equation (2.29)), and thus it cannot be generalized in any interesting way.

The intersecting membranes from (2.31) can be mapped either into the D1-F1 or into D3-D3 system (depending on the direction of smearing):

$$\begin{pmatrix} D3_{1256} \\ D3_{3456} \end{pmatrix} \overset{R_9, T_{56}}{\longleftarrow} \begin{pmatrix} M2_{12} \\ M2_{34} \end{pmatrix} \overset{R_2, T_3}{\longrightarrow} \begin{pmatrix} F1_1 \\ D1_4 \end{pmatrix}. \tag{2.33}$$

The map to D3-D3 demonstrates an existence of the holomorphic web of membranes, and the map to F1-D1 will be useful for the discussion in section 3.

Summary. Let us summarize the results of this section. We demonstrated that nontrivial brane webs preserving eight supercharges fall into two categories: they are either described by planar networks built from straight lines (this happens for (p,q) strings and five-branes), or they have profiles which are governed by holomorphic functions of two variables. After this brief overview of brane webs, we turn to construction of the geometries produced by them.

3. Geometries produced by string webs

In the previous section we reviewed the open string picture for the brane webs and such description is applicable for the webs which carry small charge. As the number of branes becomes larger, the probe approximation breaks down, but one finds another semiclassical description in terms of closed strings, and often a good quantitative description of the dynamics is given by supergravity. The next two sections will be devoted to finding the gravity solutions produced by webs of branes, and in this section we will describe the geometries corresponding to the webs of (p,q)-strings.

In the probe approximation, a generic web preserving eight supercharges contains various independent "elementary webs" located in parallel planes (see figure 1c). To construct the relevant supergravity solutions, we begin with describing a geometry produced by an "elementary web" and then find its generalization. The main advantage of this approach relies on the fact that an "elementary web" (see figure 1b) has a large bosonic symmetry, and one can derive the most general solution of IIB SUGRA which has corresponding isometries. Indeed, if an "elementary web" is stretched in x_1, x_2 directions, then it is invariant under SO(7) rotations in the orthogonal directions. To preserve these isometries, one is only allowed to excite a metric, an electric three-form, and an axion-dilaton in IIB supergravity. While it is possible to solve the SUSY variations and to find the most general geometry with SO(7) symmetry, we will pursue a different path which will also lead to geometries produced by other brane webs. Namely we begin with smearing the web in one of the orthogonal directions, thus reducing the symmetry to $\mathrm{SO}(6)\times\mathrm{U}(1)$. The advantage of this procedure is that we now have a translational isometry (the Killing spinor is not charged under the U(1) transformations), so a T duality along this direction leads to a supersymmetric configuration in IIA theory. A further lift to eleven dimensions yields a configuration of M2 branes. For example, a fundamental string $F1_1$ and a $D1_2$ brane map into the membranes with following orientations:

	M	1	2	3	4	5	6	7	8	9
$\mathrm{M2}_F$	•	•		∼						
$\mathrm{M2}_D$	∼		•	•						

(3.1)

Due to the smearings, we have translational invariance in x_3 and in the M theory direction, there is also an SO(6) rotational symmetry. Given these isometries, we find that the most general static configuration of the membranes is described by the following ansatz:

$$ds^2 = -e^{2A}dt^2 + e^{2C}(dw_1 - \chi dw_2)^2 + e^{2D}dw_2^2 + g_{MN}dX^M dX^N + e^{2B}d\Omega_5^2$$

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$$G_4 = dt \wedge W_\alpha \wedge dw^\alpha .$$

In the appendix A we found the most general supersymmetric solution which has this form and, upon translating it back to the IIB theory, we find the geometry which describes string webs:

$$\begin{aligned} ds_E^2 &= -e^{9A/4}dt^2 + e^{9A/4}h_{ab}dx^a dx^b + e^{-3A/4}(dz^2 + dy^2 + y^2 d\Omega_5^2), \\ e^{2\Phi} &= e^{3A}h_{11}^2, \quad C^{(0)} = -\frac{h_{12}}{h_{11}}, \quad B = e^{3A}h_{1a}\, dt \wedge dx^a, \quad C^{(2)} = e^{3A}h_{2a}\, dt \wedge dx^a, \\ h_{ab} &= \frac{1}{2}\partial_a\partial_b K(x_c, y), \qquad e^{-3A} = \det\, h, \qquad \frac{1}{y^5}\partial_y(y^5\partial_y K) = -2\det\, h. \end{aligned} \tag{3.2}$$

This metric was derived under assumption of $\mathrm{SO}(6)\times \mathrm{U}(1)_z$ symmetry, which can be easily relaxed: to describe an "elementary web" with SO(7) symmetry, the expression in the parenthesis should be replaced by $dy^2 + y^2 d\Omega_6^2$, and for the geometry corresponding to the most general web one expects to have an arbitrary dependence upon the seven-vector $\mathbf{y}$:

$$ds_E^2 = -e^{9A/4}dt^2 + e^{9A/4}h_{ab}dx^a dx^b + e^{-3A/4}d\mathbf{y}^2 \tag{3.3}$$
$$\tau = \frac{1}{h_{11}}(ie^{-3A/2} - h_{12}), \quad B + iC^{(2)} = e^{3A}dt \wedge (h_{1a}dx^a + ih_{2a}dx^a)$$
$$h_{ab} = \frac{1}{2}\partial_a\partial_b K(x_c, y), \qquad e^{-3A} = \det\, h, \quad \Delta_{\mathbf{y}}K = -2\det\, h. \tag{3.4}$$

These relations give a local description of the solution away form the sources and now we explain how to incorporate charged objects into this picture. In supergravity one can account for branes in two different ways: they can either be introduced as sources in the equations of motion, or their presence can be reflected in boundary conditions. We will pursue the second option.

Looking at the structure of the two-form potentials, we conclude that the y-directions should be orthogonal to the branes, so the web breaks into "elementary planes" located at specific values of $\mathbf{y}$. Moreover, in each plane the branes go along certain curves $f(x_1, x_2) = 0$. Let us introduce $w_1 = w_1(x_1, x_2)$ as a coordinate along the curve and define $w_2 = w_2(x_1, x_2)$ to be orthogonal to it. As we approach a string or a D brane, the metric component g_{tt} should go to zero, so e^{-3A} should diverge at the profile of the brane. Moreover, for consistency, the leading order of the metric should split into the longitudinal and transverse parts:

$$ds_E^2 = e^{9A/4}(-dt^2 + h_{w_1w_1}dw_1^2) + e^{-3A/4}(e^{3A}h_{w_2w_2}dw_2^2 + d\mathbf{y}^2), \tag{3.5}$$

and the expressions in parenthesis should give regular metrics. This implies that functions $h_{w_1w_1}$ and $\tilde{h}_{w_2w_2} = e^{3A}h_{w_2w_2}$ should remain finite and non-vanishing in the vicinity of the brane, so in the leading order they must be constant. Moreover, by rescaling w_a, we can set $h_{w_1w_1} = \tilde{h}_{w_2w_2} = 1$. This yields the leading contributions to the Kahler potential:

$$K = w_1^2 + 2\int^{w_2} d\tilde{w}_2 \int^{\tilde{w}_2} d\hat{w}_2 e^{-3A}. \tag{3.6}$$

Taking derivatives of this expression, we find the leading terms in the complex two-form:

$$B+iC^{(2)}=e^{3A}dt\wedge\left[dw_1\left(\frac{\partial w_1}{\partial x_1}+i\frac{\partial w_1}{\partial x_2}\right)+e^{-3A}dw_2\left(\frac{\partial w_2}{\partial x_1}+i\frac{\partial w_2}{\partial x_2}\right)\right].$$

By definition, coordinate w_2 is orthogonal to the brane, so the leading contribution to the two-form potential should not have any components along w_2. To be more precise, one should require that, being rewritten in the orthonormal frame, the w_2 component of $B+iC^{(2)}$ should either vanish or be a pure gauge. This implies that

$$\left(\frac{\partial w_2}{\partial x_1}+i\frac{\partial w_2}{\partial x_2}\right)=A \tag{3.7}$$

is a complex constant. In other words, we see that w_2 is a linear function of x_1 and x_2. Then coordinate w_1, which was defined as an orthogonal complement of w_2, must be linear as well. The rotation from (x_1,x_2) to (w_1,w_2) can be parameterized by one angle ϕ:

$$\begin{aligned} w_1&=\cos\phi\; x_1+\sin\phi\; x_2, \qquad\qquad w_2=-\sin\phi\; x_1+\cos\phi\; x_2, \\ B+iC^{(2)}&=dt\wedge\left[e^{3A+i\phi}dw_1+ie^{i\phi}dw_2\right]. \end{aligned} \tag{3.8}$$

We conclude that the string web must consist of straight elements ($x_2=\tan\phi\; x_1$) and, in a perfect agreement with (2.6), the ratio of the D1 and F1 charges is correlated with orientation of these elements. This agreement between the probe picture and supergravity solution is a manifestation of an open/closed string duality for the string webs.

The most general solution (3.3) should satisfy the equations (3.4) away from the sources which, for consistency, should be placed along straight lines in $\mathbf{y}=\mathbf{y}_i$ planes. Such string webs, along with total charge at infinity, uniquely specify the solution: since each segment of the string must satisfy (3.8), the distribution of string/brane charge at each junction is uniquely determined by geometric angles. Once proper sources are specified, one can use perturbation theory to demonstrate that solution exists and it is unique. To avoid repetition, we postpone the discussion of perturbative expansion until section 4.2, where geometries produced by membranes are analyzed. To describe perturbation theory for the string webs, one would need to pick particular profiles of the M2 branes.

4. Webs of membranes

In the previous section we discussed the geometries produced by the webs of (p,q) strings and we found a perfect agreement between the profiles of the brane probes and the allowed sources in supergravity. While providing a very nice check of the duality between open and closed strings, this agreement was somewhat mundane since the webs were constructed out of straight lines. In this section we will explore a more interesting correspondence between the membrane probes and gravity solutions in eleven dimensions, and we will see that holomorphic profiles naturally arise in both descriptions.

4.1 Structure of the solution

As reviewed in section 2, a membrane intersection preserving eight supercharges can be generalized to an M2 brane following an arbitrary holomorphic profile. The resulting configuration is still 1/4-BPS. In general we do not expect to have additional bosonic symmetries, but, just as in the case of string webs, one can consider an "elementary web of holomorphic membranes" which is located at fixed values of the transverse coordinates. To be more precise, we begin with an intersection (3.1) and remove the smearing in x_M and x_3. The resulting intersection would preserve the same supercharges as a membrane which follows an arbitrary holomorphic profile in x_M, x_1, x_2, x_3. Assuming that all such membranes are located at the same values of $x_4,\dots,x_9$, we can impose a rotational symmetry in these directions. Thus we are interested in configurations which preserve $\mathrm{U}(1)_t \times \mathrm{SO}(6)$ bosonic isometries in addition to eight supercharges. As the membrane charge becomes large, the probe analysis would break down, but at some point a geometric description would becomes semiclassical, so we need to construct supersymmetric metrics with $\mathrm{U}(1)_t \times \mathrm{SO}(6)$ isometry. While such solutions have been written before [21], that paper started with a guess inspired by [28] and checked that supersymmetry conditions were satisfied. This approach would not serve our purpose of demonstrating the duality between probe and gravitational description since we need to start with the *most general* gravity solution with given symmetries and *prove* that it has the same degrees of freedom as the probes.[7] In the appendix A we construct the most general supersymmetric solution of eleven dimensional SUGRA which has $\mathrm{U}(1)_t \times \mathrm{SO}(6)$ isometry:[8]

$$\begin{aligned} ds^2 &= -e^{2A}dt^2 + 2e^{2A}g_{a\bar b}dz^a d\bar z^b + e^{-A}(dy^2 + y^2 d\Omega_5^2) \\ G_4 &= idt\wedge d(e^{3A}g_{a\bar b}dz^a\wedge d\bar z^b), \quad g_{a\bar b} = \partial_a\bar\partial_b K. \end{aligned} \tag{4.1}$$

This geometry corresponds to an "elementary web" of the membranes, and on the probe side such elements can be freely superposed to produce a most general 1/4-BPS web. Such web consists of individual pieces located at different values of $\mathbf{y}$, and the gravity counterpart of the superposition is obvious: one should relax the requirement of SO(6) symmetry. It is easy to check the the resulting geometry,

$$\begin{aligned} ds^2 &= -e^{2A}dt^2 + 2e^{2A}g_{a\bar b}dz^a d\bar z^b + e^{-A}d\mathbf{y}_6^2, \\ G_4 &= idt\wedge d(e^{3A}g_{a\bar b}dz^a\wedge d\bar z^b), \quad g_{a\bar b} = \partial_a\bar\partial_b K, \end{aligned} \tag{4.2}$$

solves SUSY variations and equations of motion,[9] as long as Kahler potential and warp factor satisfy two relations:

$$\frac{1}{4}e^{-3A} = \partial_1\bar\partial_1 K\partial_2\bar\partial_2 K - \partial_1\bar\partial_2 K\partial_2\bar\partial_1 K, \tag{4.3}$$

$$\Delta_{\mathbf{y}}K + 2e^{-3A} = 0, \tag{4.4}$$

[7] Even solution [28], which was only partially derived, would not give an independent check of the duality: while constructing that geometry, the authors assumed that there exist complex coordinates in which the projections imposed on the Killing spinor are identical to those satisfied by the probe branes in flat space. Thus the agreement between the brane probes and gravity was the basic assumption of that derivation.

[8] To be precise, we also assumed that the field strength is purely electric.

[9] Such check would essentially follow some of the steps presented in the appendix C of [17].

but clearly (4.2) is not the most general 1/4-BPS metric with $\mathrm{U}(1)_t$ isometry.[10] Nevertheless, based on the physical picture of superposition principle and on the fact that (4.1) is the most general solution with $\mathrm{U}(1)_t \times \mathrm{SO}(6)$ isometries, we propose that (4.2) is the most general solution describing the webs of membranes. For future reference, we also write an equation which does not contain e^A:

$$\partial_1\bar{\partial}_1 K \partial_2\bar{\partial}_2 K - \partial_1\bar{\partial}_2 K \partial_2\bar{\partial}_1 K = -\frac{1}{8}\Delta_{\mathbf{y}} K. \tag{4.5}$$

As in the case of the string webs, a local solution (4.2)–(4.4) is valid away from the sources, and equations (4.3), (4.4) should be modified at the locations of the membranes. Let us demonstrate that the sources cannot be placed at arbitrary points, but rather they should follow holomorphic profiles.

Let us consider the solution in the vicinity of the sources. Looking at the three-form potential

$$C_3 = -ie^{3A} dt \wedge h_{a\bar{b}} dz^a \wedge d\bar{z}^b, \tag{4.6}$$

we conclude that $\mathbf{y}$-directions should be orthogonal to the membranes. In the four dimensional space spanned by $(z_a, \bar{z}_a)$, two directions (we call them v_1 and v_2) should be transverse to the branes as well, and two remaining directions (w_1, w_2) should parameterize the worldvolume. By definition of the longitudinal direction, in the vicinity of the membrane the warp factor e^A should not depend on w_1, w_2, i.e.

$$\partial_w \Delta_{\mathbf{y}} K|_{\mathrm{brane}} \to 0. \tag{4.7}$$

This allows us to decompose the Kahler potential into four pieces:

$$K = K_1(w_1, w_2, v_1, v_2) + K_2(v_1, v_2, \mathbf{y}) + K_{\mathrm{harm}} + K_0, \tag{4.8}$$

so that $\Delta_{\mathbf{y}} K_{\mathrm{harm}}$ and all second derivatives of K_0 vanish in the vicinity of the brane. As we argued above, the brane should be located at a particular value of vector $\mathbf{y}$, so the Kahler potential can only diverge at a point in eight dimensional space $(v_1, v_2, \mathbf{y})$. Thus we conclude that $K_1(w_1, w_2, v_1, v_2)$ must remain finite in the vicinity of the brane, so it can be replaced by the value on a brane: $K_1(w_1, w_2) \equiv K_1(w_1, w_2, v_1^{(0)}, v_2^{(0)})$. For the same reason the v_i and $\mathbf{y}$ dependence can be ignored in K_{harm} as well. This leads to conclusion that the leading contribution to Kahler potential has a separated form

$$K = K_1(w_1, w_2) + K_2(v_1, v_2, \mathbf{y}) \tag{4.9}$$

with divergent K_2 and finite K_1. This separation splits equation (4.5) into three independent pieces:[11]

$$\begin{aligned} \partial_1\bar{\partial}_1 K_1 \ \partial_2\bar{\partial}_2 K_1 - \partial_1\bar{\partial}_2 K_1 \ \partial_2\bar{\partial}_1 K_1 &= 0 \qquad (4.10) \\ \partial_1\bar{\partial}_1 K_2 \ \partial_2\bar{\partial}_2 K_2 - \partial_1\bar{\partial}_2 K_2 \ \partial_2\bar{\partial}_1 K_2 &= 0 \\ e^{-3A} = (\sigma_1)_{ab} \left[\partial_1\bar{\partial}_1 K_a \ \partial_2\bar{\partial}_2 K_b - \partial_1\bar{\partial}_2 K_a \ \partial_2\bar{\partial}_1 K_b\right] &= -\frac{1}{8}\Delta_{\mathbf{y}} K_2 \qquad (4.11) \end{aligned}$$

[10] For example, a system which consists of KK monopones with worldvolumes along 0123456, 012789M and an $M2_{012}$ brane has the same number of (super)symmetries as (4.2), but it does not fit in that ansatz.

[11] The second and third equations in this system have different degrees of the singular function K_2 and thus they have to be considered separately. The first equation is completely regular.

By introducing a new function α, the equation (4.10) can be rewritten as a linear system for K:

$$\partial_1\bar{\partial}_1 K_1 = \alpha\partial_2\bar{\partial}_1 K_1, \quad \partial_1\bar{\partial}_2 K_1 = \alpha\partial_2\bar{\partial}_2 K_1. \tag{4.12}$$

To proceed we need to solve an equation

$$(\partial_1 - \alpha\partial_2)\Phi = 0, \tag{4.13}$$

which is satisfied by both $\bar{\partial}_1 K_1$ and $\bar{\partial}_2 K_1$. Method of characteristics reduces this problem to a system of ODEs:

$$\frac{d\Phi}{ds} = 0: \qquad \frac{dz_1}{ds} = 1, \quad \frac{dz_2}{ds} = -\alpha(z_1, z_2, \bar{z}_1, \bar{z}_2). \tag{4.14}$$

which should be solved for $z_1(s)$, $z_2(s)$, while values of $\bar{z}_1$ and $\bar{z}_2$ are kept fixed. The solution is parameterized by an integration constant $\tilde{z}_2$ for the second equation:

$$z_1 = s, \quad z_2 = \beta(s, \bar{z}_1, \bar{z}_2, \tilde{z}_2): \qquad \tilde{z}_2 = \gamma(z_1, z_2, \bar{z}_1, \bar{z}_2). \tag{4.15}$$

Since the derivative $\frac{d\Phi}{ds}$ must vanish, function Φ can depend on z_1 and z_2 only through the combination $\gamma(z_1, z_2, \bar{z}_1, \bar{z}_2)$. Then, after redefining arguments of Φ, we find that the general solution of equation (4.13) is

$$\Phi = \Phi[z_2 + \Psi(z_1, \bar{z}_1, \bar{z}_2), \bar{z}_1, \bar{z}_2)]. \tag{4.16}$$

This allows us to find the first integrals of the equations (4.12):

$$\bar{\partial}_1 K_1 = f_1(z_2 + \Psi(z_1, \bar{z}_1, \bar{z}_2), \bar{z}_1, \bar{z}_2), \qquad \bar{\partial}_2 K_1 = f_2(z_2 + \Psi(z_1, \bar{z}_1, \bar{z}_2), \bar{z}_1, \bar{z}_2) \tag{4.17}$$

The mixed derivative $\bar{\partial}_1\bar{\partial}_2 K_1$ can be computed in two different ways and the consistency condition implies that[12]

$$\partial_1\bar{\partial}_1\Psi(z_1, \bar{z}_1, \bar{z}_2) = 0, \qquad \partial_1\bar{\partial}_2\Psi(z_1, \bar{z}_1, \bar{z}_2) = 0. \tag{4.18}$$

Substitution of these relations in (4.17) leads to a restriction on the form of the Kahler potential K_1:

$$\begin{array}{l} \bar{\partial}_1 K_1 = \tilde{f}_1(z_2 + \Psi(z_1), \bar{z}_1, \bar{z}_2), \\ \bar{\partial}_2 K_1 = \tilde{f}_2(z_2 + \Psi(z_1), \bar{z}_1, \bar{z}_2) \end{array} \quad \to \quad K_1 = f(z_2 + \Psi(z_1), \bar{z}_1, \bar{z}_2). \tag{4.19}$$

Since K_1 must be real (up to irrelevant (anti)holomorphic contributions), we arrive at the final expression:

$$K_1 = f(w, \bar{w}), \quad w = z_2 + \Psi(z_1). \tag{4.20}$$

[12] In the exceptional case where $f_2 = c_1 f_1 + f_3(\bar{z}_1, \bar{z}_2)$ with constant c_1, we have a weaker condition $\bar{\partial}_2\Psi = c_1\bar{\partial}_1\Psi$. Comparing other mixed derivatives, we find that, up to (anti)holomorphic functions, $K_1 = f[z_2 + \Psi(z_1, c_1\bar{z}_2 + \bar{z}_1), c_1\bar{z}_2 + \bar{z}_1]$. Then reality of K_1 implies that $K_1 = f[z_2 + \bar{c}_1 z_1, c_1\bar{z}_2 + \bar{z}_1]$. This is a particular case of (4.20).

Similarly, equation (4.11) implies that K_2 depends on holomorphic coordinate v, its conjugate $\bar{v}$, and $\mathbf{y}$:

$$K_2 = g(v, \bar{v}, \mathbf{y}). \tag{4.21}$$

To check the last equation (4.11), we pass to coordinates (w, v) and introduce a holomorphic Jacobian

$$J = \frac{D(w, v)}{D(z_1, z_2)}. \tag{4.22}$$

Then (4.11) simplifies:

$$|J|^2 \partial_w \bar{\partial}_w K_1 \; \partial_v \bar{\partial}_v K_2 = -\frac{1}{8} \Delta_{\mathbf{y}} K_2 \tag{4.23}$$

A consistency condition requires that $J = J_1(w) J_2(v)$, then, introducing reparameterizations of v and w, one can remove the Jacobians completely. Thus we demonstrated that near the brane there always exists the *unique* set of holomorphic coordinates (v, w) which brings Kahler potential to the form

$$K = \frac{1}{2} w\bar{w} + K_2(v, \bar{v}, \mathbf{y}), \qquad \partial_v \bar{\partial}_v K_2 + \frac{1}{4} \Delta_{\mathbf{y}} K_2 = 0. \tag{4.24}$$

Moreover, we showed that the location of the membrane is determined by the relations $\mathbf{y} = \mathbf{y}^{(0)}$, $v = v^{(0)}$, which implies that in the original coordinates $(z_a, \bar{z}_a)$ the brane follows a holomorphic profile:

$$v(z_1, z_2) = 0. \tag{4.25}$$

Thus we find a perfect agreement with results of the probe analysis (2.18).

Once the allowed brane profiles are specified, it is clear how to introduce sources in the system (4.2): the equation for the field strength (4.4) should be replaced by

$$\partial_a \bar{\partial}_b \left[\Delta_{\mathbf{y}} K + 8 \det(\partial \bar{\partial} K) \right] = -\sum_{i=1}^{k} Q_{(i)} \partial_a v \bar{\partial}_b \bar{v} \delta(\mathbf{y} - \mathbf{y}_{(i)}) \delta_{(2)}(v - v_{(i)}). \tag{4.26}$$

Here $\{\mathbf{y}_{(i)}, v_{(i)}(z)\}$ give the positions of the membranes and $Q_{(i)}$ specify their charges. To arrive at (4.26), one starts with an assumption that the behavior of the left hand side near the brane is not sensitive to the effects of the curvature on the worldvolume, i.e. it can be extracted from the "closeup" limit in which the membrane is flat. In this limit, the holomorphic function v can be chosen to be linear in z_1, z_2, the determinant in the left hand side of (4.26) disappears, and the entire equation (4.26) reduces to the standard Poisson equation for the "harmonic" function e^{-3A}. Making holomorphic reparameterization of v in this Poisson equation, one arrives at (4.26).

Equation similar to (4.26) was discussed in [21], where it was argued that geometries corresponding to curved membranes did not exist. This conclusion was reached by demonstrating that equation (4.26) did not admit perturbative expansion: even the second term in the series could not be defined anywhere: formally it contained infinite multiplicative constant. Physically, such divergence seems counter-intuitive: since the effects of the branes should disappear at infinity, one should be able to define a good expansion at least far away from the branes. In the next subsection we will demonstrate that the perturbation series for (4.26) indeed does exist and it is convergent, the curved membranes do produce good asymptotically-flat solutions, and erroneous statement of [21] was based on an unfortunate choice of the expansion parameter.

4.2 Solution in perturbation theory

In this subsection we will argue that any allowed distribution of membranes leads to the unique solution of equation (4.5). In particular, we will demonstrate that far away from the branes one has a well-defined perturbation theory, and we will present some evidence that analytic continuation of this perturbation leads to a good solution. This conclusion is in a sharp contradiction with the results of [21], where it was argued that neither perturbation series nor geometry exists for the curved branes.

To expand solutions of (4.26) in the powers of the charges, one should begin with specifying the solution at zeroth order in $Q_{(i)}$. Of course, the homogeneous version of (4.26) has many interesting solutions corresponding to various asymptotic behaviors of the metric, but we would be mostly interested in the case where the space is flat. Then in some coordinate system $(z_a, \bar{z}_a)$ the Kahler potential can be written as

$$K_0 = \frac{1}{2}(z_1\bar{z}_1 + z_2\bar{z}_2). \tag{4.27}$$

This starting point was also used in [21], where it was shown that perturbative expansion in powers of $Q_{(i)}$ breaks down at the second order. Before defining an improved perturbative series, let us recall the arguments of [21]. For simplicity we consider $k = 1$ and set $\mathbf{y}_{(1)} = 0$, $v_{(1)} = 0$ in (4.26).

In the first order of perturbation theory, one gets a Poisson equation:

$$\partial_a\bar{\partial}_b\left[\Delta_{\mathbf{y}}K_1 + 4(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2)K_1\right] = -Q\partial_a v\bar{\partial}_b\bar{v}\delta(\mathbf{y})\delta(v), \tag{4.28}$$

which can be easily integrated:

$$\begin{aligned}\partial_a\bar{\partial}_b K_1 &= \frac{Q}{8\Omega_9}\int\frac{d^2z'd^2\bar{z}'\partial'_a v(z')\bar{\partial}'_b\bar{v}(\bar{z}')}{[\mathbf{y}^2 + (z_c - z'_c)(\bar{z}_c - \bar{z}'_c)]^{9/2}}\delta^{(2)}(v(z') - v_0)\\ &= \frac{Q}{8\Omega_9}\int dz'_{3-a}d\bar{z}'_{3-b}\frac{1}{[\mathbf{y}^2 + (z_c - z'_c)(\bar{z}_c - \bar{z}'_c)]^{9/2}}\Bigg|_{\substack{z'_a = z^{(0)}_a(z'_{3-a})\\ \bar{z}'_b = \bar{z}^{(0)}_b(\bar{z}'_{3-b})}}\end{aligned} \tag{4.29}$$

Here $z^{(0)}_a(z'_{3-a})$ is defined as a solution of the equation $v[z^{(0)}_a(z'_{3-a}), z'_{3-a}] = 0$.

The second order of (4.26) leads to a Laplace equation

$$\Delta_{\mathbf{y}}K_2 + 4(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2)K_2 = -8\det(\partial\bar{\partial}K_1). \tag{4.30}$$

In particular, it is useful to extract the behavior of the right-hand side of this equation near the brane (where its profile was approximated by $z'_a = h_a z$):

$$
\begin{aligned}
\partial_a\bar{\partial}_b K_1 &\sim \int \frac{h_{3-a}\bar{h}_{3-b}dzd\bar{z}}{[\mathbf{y}^2+|z_c-h_c z|^2]^{9/2}} = \int_0^\infty \frac{2\pi h_{3-a}\bar{h}_{3-b}dr^2}{[\mathbf{y}^2+h^{-1}|h_2z_1-h_1z_2|^2+hr^2]^{9/2}} \\
&\sim \frac{h_{3-a}\bar{h}_{3-b}}{h[\mathbf{y}^2+h^{-1}|h_2z_1-h_1z_2|^2]^{7/2}}, \qquad h\equiv h_1\bar{h}_1+h_2\bar{h}_2 \\
\det(\partial\bar{\partial}K_1) &\sim \frac{\det(h_a\bar{h}_b)}{h^2[\mathbf{y}^2+h^{-1}|h_2z_1-h_1z_2|^2]^7}
\end{aligned}
\tag{4.31}
$$

Then, trying to solve (4.30) using Green's function, one finds a nonsensical answer even away from the sources:

$$
\begin{aligned}
K_2(z,\mathbf{y}) &= 8\int d^4w d^6x G(z,\mathbf{y}|z',\mathbf{x})\left[\det(\partial\bar{\partial}K_1)\right]_{z',\mathbf{x}} \\
&\geq G(z,\mathbf{y}|z'_a=0,\mathbf{x}=0)\int_{\mathbf{x}^2+r^2\leq\epsilon} d^4z'd^6x\left[\det(\partial\bar{\partial}K_1)\right]_{z',\mathbf{x}} = \infty.
\end{aligned}
\tag{4.32}
$$

Here r is defined as a distance from the membrane profile $v(z_1,z_2)=0$ in four-dimensional space spanned by $(z_a,\bar{z}_a)$.

This argument led authors of [21] to the conclusion that perturbation theory in charges is not well-defined and that the solutions corresponding to curved membranes do not exist. However, as we will now explain, the naive perturbation theory in Q can be modified and the resulting series gives a well-defined solution.

First we observe that the expression (4.32) gives a divergent result even far away from the sources, but infinity comes from the contribution at the location of the branes. This divergence appeared since the authors of [21] assumed that equation (4.30) was not modified near the sources (this was a consequence of making an expansion in powers of Q). A completely opposite situation is encountered for the multipole expansion: there solution is regular at infinity, but the series cannot be trusted near the sources. Physically it is clear that one is interested in the multipole rather than Q-expansion, but these two series can be easily confused. For example, looking at a function

$$
f=\frac{1}{1+\frac{Q}{r}}=1-\frac{Q}{r}+\frac{Q^2}{r^2}+\dots, \tag{4.33}
$$

one may conclude that the large–r and small–Q expansions look the same,[13] but it is important to keep in mind that the series should not be taken seriously at small values of r, and one is allowed to modify any *perturbative* equation for f at $r=0$ to ensure the correct large-distance behavior.

To illustrate such modification, we go back to the equation (4.30) and add an extra term to its right-hand side:

$$
\Delta_{\mathbf{y}}K_2+4(\partial_1\bar{\partial}_1+\partial_2\bar{\partial}_2)K_2=-8\det(\partial\bar{\partial}K_1)+Q_0^{(2)}\delta(\mathbf{y})\delta^{(2)}(v). \tag{4.34}
$$

[13]This is expected since Q/r is the only dimensionless parameter in the problem.

The value of $Q_0^{(2)}$ can be fixed by eliminating the leading divergent contribution to K_2 at infinity (i.e. by requiring the correction to the membrane charge to vanish). To show that such $Q_0^{(2)}$ can always be chosen, we surround the membrane by a shell with radius ϵ and solve equation (4.34) by restricting integration in (4.32) to the exterior of the shell and by adding a term proportional to $Q_0^{(2)}$:

$$\begin{aligned} K_2^\epsilon(z,\mathbf{y}) = 8\int_{r>\epsilon} & d^4z' d^6x G(z,\mathbf{y}|z',\mathbf{x})\left[\det(\partial\bar{\partial}K_1)\right]_{z',\mathbf{x}} \\ & -Q_0^{(2)}\int d^4z' G(z,\mathbf{y}|z',\mathbf{0})\delta_{(2)}(v[z']) \end{aligned} \tag{4.35}$$

The first term in this equation diverges as ϵ goes to zero, and the leading pole has the same functional dependence on $(z,\mathbf{y})$ as the second term. Thus by taking $Q_0^{(2)} = \frac{\tilde{Q}_0^{(2)}}{\epsilon^p}$, one can eliminate the leading divergence in $K_2^\epsilon(z,\mathbf{y})$. Notice that leading order in $1/\epsilon$ is also the leading contribution in $1/r$ expansion (it gives the charge of the membrane measured from infinity), so it is convenient to shift $Q_0^{(2)}$ by an ϵ-independent term and to require

$$\frac{K_2^\epsilon(z,\mathbf{y})}{\int d^4z' G(z,\mathbf{y}|z',\mathbf{0})\delta^{(2)}(v[z'])} = O(r^{-1}). \tag{4.36}$$

Going back to equation (4.35), we observe that the new leading term in ϵ-expansion behaves like a potential of a "dipole membrane" far away from the sources, and it can be canceled by adding an appropriate local counterterm to the right-hand side of (4.34). Acting in a similar fashion, one can modify (4.34) by adding a series of extra "multipole" sources which are localized on the membrane:

$$\begin{aligned} K_2^\epsilon(z,\mathbf{y}) = 8\int_{r>\epsilon} & d^4z' d^6x G(z,\mathbf{y}|z',\mathbf{x})\left[\det(\partial\bar{\partial}K_1)\right]_{z',\mathbf{x}} \\ & -\sum_k Q_k^{(2)}\int d^4z' \Delta_{\mathbf{y}}^k G(z,\mathbf{y}|z',\mathbf{0})\delta_{(2)}(v[z']), \end{aligned} \tag{4.37}$$

and which make K_2^ϵ finite. Of course, once ϵ is sent to zero, the coefficients in front of these sources would diverge, but the resulting function $K_2^{\epsilon=0}$ is well-defined and it satisfies equation (4.30) away form the branes. Then we will *define* $K_2 \equiv K_2^{\epsilon=0}$ as a second term in the perturbation series in Q:

$$K = K_0 + QK_1 + Q^2K_2 + \dots \tag{4.38}$$

The same procedure can be repeated for the higher orders in perturbation series. Moreover, by choosing the finite contributions to $Q_p^{(m)}$, we can also ensure that $K_n \ll K_{n-1}$ far away from the branes and that the first p terms in the series (4.38) correctly reproduce the first p multipole moments of the brane configuration (the moments of e^{-3A} can be extracted from the probe analysis, then the moments of K are found by integrating (4.4)). Notice that the series (4.38) should be viewed as $1/r$ rather than Q-expansion. To make such series possible, we had to modify the sources at the location of the brane, but since the series (4.38) breaks down long before these points (e.g. the expansion (4.33) breaks down at $r = Q$, while the sources are located at $r = 0$), the vicinity of the branes requires special consideration.

To summarize, we showed that, while the series in powers of Q does not make sense [21], the large r-expansion is well-defined, but it requires introduction of new sources at the location of the brane. The $1/r$-expansion is expected to have a nonzero radius of convergence, and the Kahler potential in the entire space can be constructed by analytic continuation (equation (4.33) gives the simplest example). Moreover, by construction, K satisfies a differential equation

$$\Delta_{\mathbf{y}} K + 8\det(\partial\bar{\partial}K) = -\frac{Q}{2\pi}\delta(\mathbf{y})\log(v\bar{v}) \tag{4.39}$$

away from the sources and all multipole moments of K match those of the brane configuration. To prove the relation (4.26), we need to demonstrate that (4.39) holds everywhere. Unfortunately, such proof cannot be performed in perturbation theory (since an analytic continuation was involved), so a better understanding of the nonlinear equation (4.39) is required. However, it seems plausible that, for any given (infinite) set of multipole moments, there exists only one solution of (4.39) away from the sources. Then it seems natural to *define* such solution as a geometry produced by the curved membrane, and, if such solution has extra terms on the right-hand side of (4.26),[14] one should take it as a sign of breakdown in (4.26): after all, the source terms in that equation came from the simplest generalization of branes in flat space.[15] However, we believe that the source terms in (4.26) are correct, and that analytic continuation of (4.38) satisfies equation (4.39) everywhere. The argument was outlined in section 4.1: near the brane one expects the curvature effects to become irrelevant, then (4.26) comes from the source term for the flat brane. It would be very nice to find a rigorous proof of this proposal.

To summarize, in this subsection we demonstrated that, while the solutions of equation (4.26) cannot be written as naive series in powers of Q [21], they admit a well-defined multipole expansion which converges far away from the branes and which can be extended to the entire space away from the brane profiles. By construction, our solution shares an infinite set of quantum numbers with probe configuration, so we declared that our geometry should describe an appropriate stack of membranes. We also conjectured that an analytic continuation of the series satisfies a nonlinear equation with sources (4.26). While some heuristic evidence for this proposal was given, it would be nice to find a more rigorous proof of the conjecture. It is clear that, while we only discussed a single stack of membranes, the results also hold for an arbitrary number of stacks.

5. Webs of five- and three-branes

5.1 Summary of the solutions

As we discussed in section 2, there are two ways of constructing interesting webs of five-branes in IIB string theory. Both configurations can be obtained by the chain of dualities

[14] Notice that our construction guarantees that such terms may arise only on the surface of the membrane.

[15] The same sources were derived in [21] from the probe analysis, but there it was assumed that geometry did not backreact to deform the profiles. This assumption is essentially equivalent to taking a locally-flat approximation for the branes which led to (4.26).

from the system which we already discussed. For example, starting with a web of membranes, one can perform the following transformations:

$$\begin{pmatrix} M2_{45} \\ M2_{67} \end{pmatrix} \to \begin{pmatrix} D2_{45} \\ D2_{67} \end{pmatrix} \to \begin{pmatrix} D5_{12345} \\ D5_{12367} \end{pmatrix}_1 \tag{5.1}$$

$$\downarrow$$

$$\begin{pmatrix} D4_{1245} \\ D4_{1267} \end{pmatrix} \to \begin{pmatrix} M5_{(10)1245} \\ M5_{(10)1267} \end{pmatrix}_2 \to \begin{pmatrix} NS5_{12345} \\ D4_{1237} \end{pmatrix} \to \begin{pmatrix} NS5_{12345} \\ D5_{12347} \end{pmatrix}_3$$

Tracing the metrics through the chain of dualities, we find the geometries produced by the webs of five-branes:

1. *Webs of D5 branes.* Performing the first two steps in (5.1), we find the geometry describing a web of D5 branes:

$$ds^2_{\rm IIB} = e^{3A/2}\left[-dt^2 + d\mathbf{x}_3^2 + 2g_{a\bar b}dz^a d\bar z^b\right] + e^{-3A/2}d\mathbf{y}_2^2 \tag{5.2}$$

$$F_7 = idt \wedge d(e^{3A}g_{a\bar b}dz^a \wedge d\bar z^b) \wedge d^3\mathbf{x}, \quad e^{2\Phi} = e^{3A}, \quad g_{a\bar b} = \partial_a\bar\partial_b K. \tag{5.3}$$

For this solution Kahler potential should be a function of z_a, $\bar z_a$ and $\mathbf{y}_2$ and away for the sources it should satisfy differential equation:

$$\partial_a\bar\partial_b\left[\Delta_{\mathbf{y}}K + 8\det(\partial\bar\partial K)\right] = 0. \tag{5.4}$$

Repeating the analysis presented in the previous section, one can demonstrate that D5 branes can be added to this solution, but for consistency of IIB supergravity they should follow holomorphic profiles $v(z_1, z_2) = 0$. This conclusion is in a perfect agrees with results of the probe analysis. In the presence of sources equation (5.4) should be replaced by (4.26).

2. *Webs of M5 branes.* Going back to the membrane web and performing dualities outlined in (5.1), we arrive at the geometry produced by a web of M5 branes:

$$ds^2 = e^{A}\left[-dt^2 + d\mathbf{x}_3^2 + 2g_{a\bar b}dz^a d\bar z^b\right] + e^{-2A}d\mathbf{y}_3^2 \tag{5.5}$$

$$F_7 = idt \wedge d(e^{3A}g_{a\bar b}dz^a \wedge d\bar z^b) \wedge d^3\mathbf{x}.$$

The regularity conditions work in the same way as before. This ansatz has been previously discussed in [28].

3. *(p,q)-fivebranes.* The web of (p, q) fivebranes is a magnetic counterpart of the string web. To find a geometry produced by it, we need to assume two translational isometries in four dimensional subspace spanned by $(z_a, \bar z_a)$ in (5.5), then applying the arguments presented in the appendix A.4, one can show that $z_a = r_a + iw_a$, and that nothing depends on w_a. We can then reduce this system on w_1 and perform a

T duality along w_2:

$$ds^2_{\rm IIB,E} = e^{3A/4}\left[-dt^2 + d\mathbf{x}_3^2 + dz^2 + h_{ab}dr^a dr^b\right] + e^{-9A/4}d\mathbf{y}_3^2, \tag{5.6}$$

$$e^{2\Phi} = h_{11}^2 e^{3A}, \quad C_0 = -\frac{h_{12}}{h_{11}}, \quad h_{ab} = \frac{1}{2}\partial_a\partial_b K, \quad e^{-3A} = \det\, h. \tag{5.7}$$

To evaluate the two-form potentials, one needs to perform an electric-magnetic duality in (5.5), and we will not do this here.

As in the case of the string webs, one can see that the branes must go along straight lines in (r_1, r_2) directions and that the orientation of the elements of the web is correlated with the amount of D5/NS5 charge (see discussion which led to equation (3.8)).

4. *Webs of D3 branes.* The last interesting system in (2.4) is a web of D3 branes. To construct the appropriate supergravity solution we need to apply one T duality to the D2-D2 system in (5.1):

$$\begin{aligned} ds^2_{\rm IIB} &= e^{3A/2}\left[d\mathbf{x}_{1,1}^2 + 2g_{a\bar b}dz^a d\bar z^b\right] + e^{-3A/2}d\mathbf{y}_4^2 \\ F_5 &= -id^2x\wedge d(e^{3A}g_{a\bar b}dz^a\wedge d\bar z^b), \quad e^{2\Phi} = 1. \end{aligned} \tag{5.8}$$

The regularity conditions again lead to the holomorphic profiles of the branes ($v(z_a, \bar z_a) = 0$), and the equations of motion with sources become

$$\begin{aligned} e^{-3A} &= 4\det(\partial\bar\partial K), \\ \partial_a\bar\partial_b\left[\Delta_{\mathbf{y}}K + 8\det(\partial\bar\partial K)\right] &= -\sum_{i=1}^{k} Q_{(i)}\partial_a v\bar\partial_b\bar v\delta(\mathbf{y}-\mathbf{y}_{(i)})\delta(v - v_{(i)}). \end{aligned} \tag{5.9}$$

It is interesting to compare this solution with an alternative description of metrics produced by D3 branes which was presented in [17].

5.2 Alternative description

Let us briefly review the construction of [17]. In that paper it was shown that all metrics produced by 1/4-BPS webs of D3 branes can be written in terms of one function $F(\mathbf{x},\mathbf{y},w)$:[16]

$$\begin{aligned} ds^2 &= H^{-1}ds^2_{1,1} + Hd\mathbf{y}_4^2 \\ &\quad + H^{-1}\left\{H^2h^{-1}\left[(\partial_w F dw + \partial_{\mathbf{y}}Fd\mathbf{y})^2 + (du + \epsilon_{ij}\partial_i F dx^j)^2\right] + hd\mathbf{x}_2^2\right\}, \end{aligned} \tag{5.10}$$

$$\partial_w h = -\Delta_{\mathbf{x}}F|_{y,w}, \quad H^{-2} = h^{-1}\partial_w F|_{\mathbf{x},\mathbf{y}}. \tag{5.11}$$

Function F satisfies a system of differential equations:

$$\partial_F H^2 + (\Delta_{\mathbf{y}}w)_{x,F} = 0, \quad \Delta_{\mathbf{y}}e^{-2\phi} + \Delta_{\mathbf{x}}H^2 + \Delta_{\mathbf{y}}(\partial_w F\partial_{x_i}w\partial_{x_i}w)|_{x,F} = 0,$$

[16] Equation (5.10) was derived in [17] assuming translational invariance in u direction. To compare (5.10) with [17], one should make two replacements: $e^{-H-\phi}\to H$, $e^{-2\phi}\to h$ in equation (4.47) of that paper. The expression for F_5 can be found in [17].

which allow to determine F uniquely once the sources are specified (see [17]). In particular, supergravity equations are consistent if and only if the branes follow harmonic profiles:

$$\{F = p(\mathbf{x}), \mathbf{y} = \mathbf{y}_{(0)}\}, \quad \{\mathbf{x} = \mathbf{x}_{(0)}, \mathbf{y} = \mathbf{y}_{(0)}\}. \tag{5.12}$$

To compare with analysis of the previous subsection, we need to introduce complex coordinates in the metric appearing in the curly brackets in (5.10):

$$\begin{aligned} ds_4^2 &= H^2 h^{-1}\left[(dF - \partial_{\mathbf{x}} F d\mathbf{x})^2 + (du + \epsilon_{ij}\partial_i F dx^j)^2\right] + hd\mathbf{x}_2^2 \\ &= H^2 h^{-1}\left[\left(dF + \frac{h}{H^2}\partial_{\mathbf{x}} w d\mathbf{x}\right)^2 + \left(du - \frac{h}{H^2}\epsilon_{ij}\partial_i w dx^j\right)^2\right] + hd\mathbf{x}_2^2 \\ &= H^2 h^{-1} dW d\bar{W} + 2(\bar{\partial}_z w\ dW d\bar{z} + cc) + h(4H^{-2}|\partial_z w|^2 + 1) dz d\bar{z}. \end{aligned} \tag{5.13}$$

The holomorphic coordinates turned out to be $z = x_1 + ix_2$, $W = F + iu$. In the process of deriving (5.13) we used the following relations:

$$\partial_x F = -\partial_w F \partial_x w = -H^{-2} h \partial_x w, \quad \epsilon_{ij}\partial_i w dx^j = i(\partial_z w dz + \partial_{\bar{z}} w d\bar{z}) \tag{5.14}$$

The Kahler potential corresponding to the metric (5.13) is

$$K = 2\int dF w : \partial_W \bar{\partial}_W K = \frac{1}{4}\partial_F^2 K = \frac{1}{2} H^2 h^{-1}, \quad \partial_z \bar{\partial}_W K = \partial_z w,$$
$$\partial_z \bar{\partial}_z K = 2\int dF \Delta_{\mathbf{x}} w.$$

Thus we conclude that the metric (5.10) can be rewritten as

$$ds^2 = H^{-1} ds_{1,1}^2 + H d\mathbf{y}_4^2 + 2H^{-1} g_{a\bar{b}} dz^a d\bar{z}^b. \tag{5.15}$$

Moreover, one finds that

$$\det g_{a\bar{b}} = \frac{1}{4} H^2, \tag{5.16}$$

in a perfect agreement with (4.3). Thus we have shown that some of the solutions derived in [17] fit nicely into the metric ansatz (5.8).

6. 1/4-BPS bubbling solutions of IIB SUGRA

In the last three sections we discussed various brane intersections preserving eight supercharges. However, while deriving the gravity solutions, we made important assumptions that the geometries were static, and that Killing spinor did not depend on the time coordinate. The first assumption was motivated by the probe analysis of section 2, where it was shown that brane webs in *flat* space formed static configurations. The second assumption originated from the fact that supersymmetric geometries with flat asymptotics must have a translational Killing vector. For the solutions with more general asymptotics, supersymmetry only requires an existence of a time-like (or light-like) Killing vector which does not have to be hypersurface-orthogonal (i.e. "rotating geometries" are allowed). Moreover, the Killing spinors might be charged under time translations.[17] The simplest example of the "rotational time-like symmetry" is a translation along global time in AdS space, but such Killing vector is still hypersurface-orthogonal. On the other hand, by shifting one of the angular coordinates, one can introduce a new "time" on AdS_5 which mixes with other coordinates:

$$ds_5^2 = \cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\left[d\theta^2 + \cos^2\theta(d\phi + d\tau)^2 + \sin^2\theta d\psi^2\right],$$

but half of the Killing spinors is neutral under the shift symmetry in τ (while the other half is charged). These simple examples show that we can independently relax the requirements of staticity and neutrality of the Killing spinors.

While study of general 1/4 geometries on spaces with arbitrary asymptotics goes beyond the scope of this paper, in this section we will discuss the geometries which asymptote to $AdS_5 \times S^5$. Such configurations are important for constructing the closed-string description of various states in $\mathcal{N}=4$ super-Yang-Mills. Let us begin with recalling the situation for the 1/2-BPS states. On the field theory side, they are described by excitations of matrix harmonic oscillator, in particular semiclassical states are represented by the droplets of incompressible Fermi fluid [29]. On the bulk side, one has three regimes which are well-understood. The operators with small conformal dimension are represented by perturbative gravitons on $AdS_5 \times S^5$, the operators with $\Delta \sim N$ have semiclassical description in terms of D branes [30], and semiclassical states with $\Delta \sim N^2$ correspond to regular geometries [20]. The map presented in [20] is very explicit: the boundary conditions for the metrics are identified with distributions of the fermionic droplets in the phase space. It would be very nice to have a similar picture for the states preserving a smaller amount of SUSY.

In the 1/4 BPS case, the field theory side is understood, and states are constructed from two commuting matrices [31]. On the bulk side, we again expect to have gravitons for small values of Δ and branes ("giant gravitons") for $\Delta \sim N$. In this case the giant gravitons are parameterized by holomorphic surfaces [24] and we review this construction in section 6.4. As dimension of the operator becomes of order N^2, the geometric description takes over and the local structure of the metric was described in [19]. However, to compare with field theory, one also needs to specify the allowed boundary conditions and, unfortunately, this ingredient has been missing. The main goal of this section is to clarify the admissible boundary conditions for the geometries and to show that they are consistent with expectation coming from both field theory and brane probe analysis.

6.1 Local description

Let us recall construction of bubbling geometries preserving 8 supercharges. On the field theory side, 1/4-BPS states can be represented as "words" constructed out of two com-

[17] Such symmetry is usually called "rotational" to distinguish it from the "translational" symmetry where the spinors are neutral.

muting matrices[18] X, Y. Starting with elementary building blocks

$$\mathrm{tr}(X^{n_1}Y^{n_2}), \tag{6.1}$$

one can write the most general state by combining various products of traces. Since matrix Z does not appear in the wavefunction, all 1/4-BPS states are invariant under U(1) rotation $Z \to e^{i\psi}Z$. In the context of AdS/CFT one is interested in the field theory defined on $R_\tau \times S^3$, and it turns out that for a state (6.1) to preserve SUSY, it can only contain zero modes of X and Y on S^3. This implies that 1/4-BPS states have an $\mathrm{SO}(4) \times \mathrm{U}(1)$ symmetry which should also be preserved by their gravity duals. Moreover, the states constructed from building blocks (6.1) are also symmetric under exchange of Z and $\bar{Z}$. This implies that the dual metric should be invariant under Z_2 symmetry $\psi \to -\psi$, and this angle should not mix with the remaining coordinates.

The local supersymmetric geometries with $\mathrm{SO}(4) \times \mathrm{U}(1)$ isometries have been constructed in [22, 19]:[19]

$$\begin{aligned}
ds_{10}^2 &= -h^{-2}(dt+\omega)^2 + h^2\left[\frac{2}{Z+\frac{1}{2}}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b + dy^2\right] + y(e^G d\Omega_3^2 + e^{-G}d\psi^2)\\
F_5 &= \left\{-d[y^2 e^{2G}(dt+\omega)] - y^2 d\omega + 2i\partial\bar{\partial}K\right\} \wedge d\Omega_3 + \mathrm{dual}\\
h^{-2} &= 2y\cosh G, \quad Z \equiv \frac{1}{2}\tanh G = -\frac{1}{2}y\partial_y(y^{-1}\partial_y K)\\
\det h_{a\bar{b}} &= y\left(Z+\frac{1}{2}\right)\exp\left[y^{-1}\partial_y K\right] W(z)\bar{W}(\bar{z})\\
d\omega &= \frac{i}{y}\left(\partial_a\bar{\partial}_b\partial_y K\; dz^a d\bar{z}^b - \partial_a Z\; dz^a dy + \bar{\partial}_a Z d\bar{z}_a dy\right) = \frac{i}{2}d\left[\frac{1}{y}\bar{\partial}\partial_y K - \frac{1}{y}\partial\partial_y K\right]
\end{aligned} \tag{6.2}$$

Using reparameterizations, we can impose the gauge $W = \frac{1}{2}$. Since y coordinate is equal to the product of two warp-factors (one for S^3 and one for S^1), we should impose certain boundary conditions to avoid singularities at $y = 0$ hypersurface. This issue will be addressed in section 6.5, here we just observe that when S^3 contracts to zero size (while $g_{\psi\psi}$ remains finite) function Z is necessarily equal to $-\frac{1}{2}$, while $Z = \frac{1}{2}$ when ψ-circle collapses. This situation is completely analogous to the picture for the 1/2-BPS states [20], but, as we will see later, regularity for the 1/4-BPS case leads to some additional requirements.

6.2 Examples

Before discussing the boundary conditions for an arbitrary 1/4-BPS solution, it might be useful to consider some examples. In particular, the simplest example of geometry which fits into the ansatz (6.2) is $AdS_5 \times S^5$, although coordinates used in (6.2) are not very

[18] We are using the standard notation for adjoint scalars in $\mathcal{N} = 4$ theory: six matrices $\Phi_1, \dots \Phi_6$ are often combined into three holomorphic objects $X = \Phi_1 + i\Phi_2$, $Y = \Phi_3 + i\Phi_4$, $Z = \Phi_5 + i\Phi_6$.

[19] Solution (6.2) is written in the notation introduced in [23]. More general family of 1/4-BPS geometries with $\mathrm{SO}(4) \times \mathrm{U}(1)$ isometries was constructed in [32], but due to the mixing between U(1) direction and other coordinates, these solutions break Z_2 symmetry, so they are not relevant for describing the states (6.1), and we will not discuss these metrics further.

standard. Another important example is given by a family of 1/2-BPS solutions which have an enhanced SO(4) $\times$ SO(4) symmetry [20]. In this subsection we will embed these two solutions into (6.2).

6.2.1 AdS$_5\times$S^5 as a 1/4-BPS state in a theory on $R\times S^3$

Starting from the standard metric on AdS$_5\times$S^5:

$$\begin{aligned} ds^2 &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 \\ &\quad + \sin^2\theta d\psi^2 + d\theta^2 + \cos^2\theta\left[\cos^2\alpha d\phi_1^2 + \sin^2\alpha d\phi_2^2 + d\alpha^2\right], \end{aligned} \tag{6.3}$$

it is easy to find the change of coordinates which leads to (6.2). The appropriate map was derived in [23], and here we just write down the answer which will be used later on. Following [23], we introduce complex coordinates

$$z_1 = r\cos\alpha e^{i(\phi_1+t)}, \quad z_2 = r\sin\alpha e^{i(\phi_2+t)}, \quad r = \cosh\rho\cos\theta. \tag{6.4}$$

By construction, the subspace spanned by $(z_a,\bar{z}_a)$ is orthogonal to $y=\sinh\rho\sin\theta$. Then direct computations lead to expressions for the Kahler potential:

$$\begin{aligned} K &= \frac{1}{2}\left[\Psi - \log\Psi - y^2\log(\Psi - r^2) + y^2\log y - y^2\right], \\ \Psi &\equiv \frac{1}{2}(r^2+y^2+1) + \sqrt{\frac{1}{4}(r^2+y^2-1)^2 + y^2}, \end{aligned} \tag{6.5}$$

for the one-form ω, and for the function Z:

$$\omega = \frac{h^2}{\cosh^2\rho}\mathrm{Im}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2), \quad Z = \frac{h^2}{2}(r^2+y^2-1), \quad h^{-2} = \sinh^2\rho + \sin^2\theta. \tag{6.6}$$

It is interesting to look at the boundary conditions for function Z at $y=0$. Since y is a product of two functions, the hypersurface $y=0$ is divided into two regions:

$$\begin{aligned} &\rho = 0: \qquad && r = \cos\theta \le 1, \qquad && Z = -\frac{1}{2}, \\ &\theta = 0: \qquad && r = \cosh\rho \ge 1, \qquad && Z = \frac{1}{2}, \end{aligned}$$

and the boundary between the regions is a three-sphere in $\mathbf{C}^2$:

$$z_1\bar{z}_1 + z_2\bar{z}_2 = 1. \tag{6.7}$$

6.2.2 AdS$_5\times$S^5 as a 1/4-BPS geometry with SO(4) R-symmetry

While we will mostly be interested in the representation (6.5), there is an alternative way of embedding $AdS_5\times S^5$ into the general 1/4-BPS ansatz (6.2). Unlike (6.5) which preserves the sphere from AdS_5 (this fact makes (6.5) useful for studying normalizable states in SYM on $R\times S^3$), the other representation breaks space-time rotational invariance, while keeping a large part of the R-symmetry group.

To find such alternative embedding, we begin with rewriting the metric on $AdS_5 \times S^5$:

$$\begin{aligned} ds^2 &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left[\sin^2\alpha d\psi^2 + \cos^2\alpha d\tilde{\beta}^2 + d\alpha^2\right] \\ &\quad + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\Omega_3^2 \end{aligned} \tag{6.8}$$

Looking at the warp factors, we can easily extract y and e^G:

$$y = \sin\theta\sinh\rho\sin\alpha, \quad e^G = \frac{\sin\theta}{\sinh\rho\sin\alpha}, \quad h^{-2} = \sinh^2\rho\sin^2\alpha + \sin^2\theta \tag{6.9}$$

To put the metric (6.8) in the form (6.2), one need to shift the angular variables ($\tilde{\beta} = \beta + t$, $\tilde{\phi} = \phi + t$), this will ensure that in the new coordinates $g_{tt} = -h^{-2}$. Such shift also introduces mixings between time and angular coordinates and one can easily read off the relevant one-form:

$$\omega = -h^2(\sinh^2\rho\cos^2\alpha d\beta + \cos^2\theta d\phi) \tag{6.10}$$

The Kahler base is parameterized by the two angles (β, ϕ) and two more coordinates which should be orthogonal to y. Starting with three-dimensional space spanned by (ρ, α, θ), one can use the metric (6.8) to construct a subspace orthogonal to y, and to show that it can be parameterized by

$$x_1 = \cosh\rho\cos\theta, \quad x_2 = \tanh\rho\cos\alpha. \tag{6.11}$$

It is now easy to invert the relations between (ρ, α, θ) and (x_1, x_2, y):

$$\begin{aligned} \cosh\rho &= \frac{1}{\sqrt{2(1-x_2^2)}}\left[1 + r^2 + y^2 + \sqrt{(1+r^2+y^2)^2 - 4r^2}\right]^{1/2}, \\ \cos\theta &= \frac{1}{\sqrt{2}}\left[1 + r^2 + y^2 - \sqrt{(1+r^2+y^2)^2 - 4r^2}\right]^{1/2}, \quad r = x_1\sqrt{1-x_2^2} \\ \sin\alpha &= \frac{\sqrt{1-x_2^2}}{\sqrt{2}}\left[\frac{-2y^2 + x_2^2(r^2+y^2-1+\sqrt{(1+r^2+y^2)^2-4r^2})}{x_2^2(r^2+y^2-1)+x_2^4-y^2}\right]^{1/2}. \end{aligned} \tag{6.12}$$

Using these expressions, we can rewrite e^G and Z as functions of (x_1, x_2, y), and the expression for Z turns out to be especially simple:

$$Z = -\frac{1}{2}\frac{r^2 + y^2 - 1}{\sqrt{(1+r^2+y^2)^2 - 4r^2}} \tag{6.13}$$

As expected, at the $y = 0$ surface this function takes only two values: $Z = \pm\frac{1}{2}$. Since Z is related to the y-derivatives of the Kahler potential by one of the equations in (6.2), we can extract the expression for K:

$$\begin{aligned} K &= \frac{1}{4}\Bigg[-R + (y^2+2)\log(1+r^2+y^2+R) \\ &\quad - y^2\log\left\{2\frac{(r^2-1)R + R^2 - y^2(1+r^2+y^2)}{y^2(r^2-1)^2}\right\}\Bigg] + K_0 + y^2K_1 \end{aligned} \tag{6.14}$$

$$R \equiv \sqrt{(1+r^2+y^2)^2 - 4r^2} \tag{6.15}$$

The "integration constants" K_0 and K_1, which may depend upon the coordinates on the base, will be evaluated below. Even though a significant part of the Kahler potential has been determined, it cannot be used to calculate the metric unless the proper complex coordinates are found. Comparing the structure of (6.10) and (6.2) and noticing that Kahler potential does not depend on the angular variables, one concludes that such coordinates must have the following form

$$z_1 = r_1 e^{i\phi}, \quad z_2 = r_2 e^{i\beta} : \quad \omega = \frac{1}{2y}\partial_y \left[r_1 \partial_1 K d\phi + r_2 \partial_2 K d\beta \right] \tag{6.16}$$

Comparing this with (6.10) and performing y-integration, we find the expression for the derivatives (up to y-independent functions):

$$r_1 \partial_1 K = \frac{1}{2}(1 + y^2 - R) + \tilde{K}_2, \quad r_2 \partial_2 K = -\frac{x_2^2 (y^2 + R)}{2(1 - x_2^2)} + \tilde{K}_1 \tag{6.17}$$

For these relations to be consistent with (6.14), we must set

$$r_1 = r\sqrt{1 - x_2^2}, \quad r_2 = x_2, \quad K_1 = \frac{1}{2}\log\frac{r(1 - x_2^2)}{r^2 - 1} \tag{6.18}$$

One can also express $\tilde{K}_1$ and $\tilde{K}_2$ in terms of derivatives of K_0, but we will not discuss this further.

To complete the expression for the Kahler potential we still need to evaluate K_0, and the easiest way to do so is to look at the metric on the $y = 0$ surface. In particular, a restriction of the metric on the base to the two-dimensional subspace spanned by (t, β, ϕ) is given by

$$\frac{\frac{\partial}{\partial \log r_a}\frac{\partial}{\partial \log r_b}K}{(2Z + 1)} d\phi_a d\phi_b = \left[h_\rho^2 c_\alpha^2 (s h_\rho^2 + s_\theta^2) d\phi_1^2 + c_\theta^2 (s h_\rho^2 s_\alpha^2 + 1) d\phi_2^2 + 2 s h_\rho^2 c_\alpha^2 c_\theta^2 d\phi_1 d\phi_2 \right]$$

and looking at this relation at $y = 0$, we find the expression

$$K_0 = \frac{r^2}{4} - \log r. \tag{6.19}$$

One can check that the resulting Kahler potential satisfies the Monge-Ampere equation.

Let us summarize the data for the $AdS_5 \times S^5$. The four-dimensional base is parameterized by the complex coordinates (6.16) and the Kahler potential is given by (6.14) with

$$K_0 + y^2 K_1 = \frac{y^2}{2}\log\frac{r(1 - x_2^2)}{r^2 - 1} + \frac{r^2}{4} - \log r, \quad r = \frac{r_1}{\sqrt{1 - r_2^2}}, \quad r_2 = x_2 \tag{6.20}$$

As already mentioned, function (6.13) takes values $\pm\frac{1}{2}$ on the hyperplane $y = 0$, and regions with different signs of Z are separated by the surface

$$r^2 = 1 : \qquad z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1. \tag{6.21}$$

Although this relation looks the same as (6.7), the base spaces for two descriptions of $AdS_5 \times S^5$ are very different. While (6.7) represented a sphere carved out of $\mathbf{C}^4$, in the present case z coordinates cover only a cylinder $|z_2| < 1$. Notice that the infinity of AdS space is mapped into the region $|z_1| = \infty$ and into the boundary of the z_2-circle ($|z_2| = 1$). The boundary conditions for two representations of $AdS_5 \times S^5$ are depicted in figure 2.

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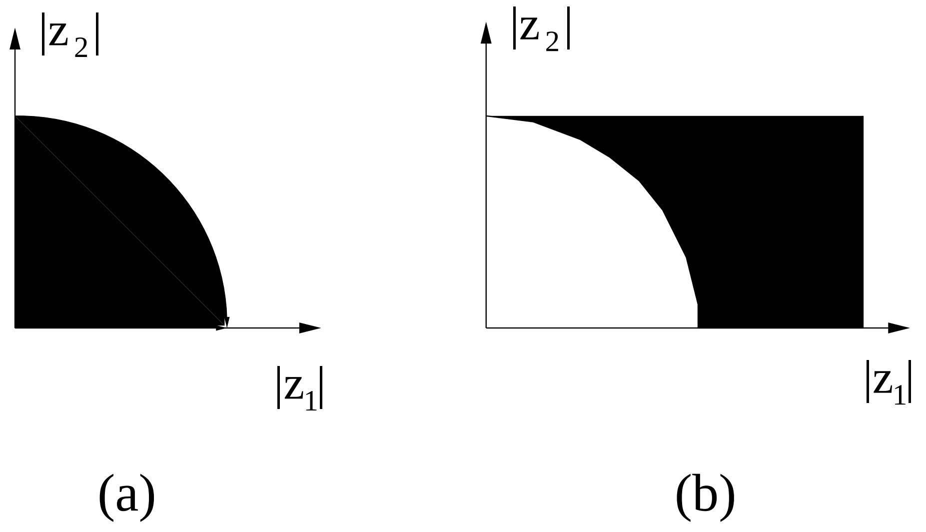

Figure 2: Boundary conditions corresponding to two embeddings of $AdS_5 \times S^5$: (a) spacial sphere S^3 is preserved, (b) SO(4)-part of the R-symmetry group is unbroken.

6.2.3 1/2-BPS bubbling solutions

While $AdS_5 \times S^5$ (along with pp-wave) presents the simplest example of a metric which can be embedded in the ansatz (6.2), there is also a more general class of known solutions covered by (6.2). Unlike a generic metric (6.2), these geometries preserve 16 rather than 8 supercharges, and they have a very explicit description in terms of solutions of the Laplace equation [20]. Thus it is useful to embed these geometries into the general ansatz (6.2).

We begin with recalling some basic facts about the 1/2-BPS geometries constructed in [20]. The metric and fluxes are parameterized by one function which depends on three variables: $\tilde{Z}(x, z, \bar{z})$, and, with slight notational modifications, the solution of [20] reads:

$$\begin{aligned} ds^2 &= -\tilde{h}^{-2}(dt+V)^2 + \tilde{h}^2(dx^2 + dzd\bar{z}) + xe^{H}d\Omega_3^2 + xe^{-H}d\tilde{\Omega}_3^2, \\ F_5 &= -\frac{1}{4}d\left[x^2 e^{2H}(dt+V) + x^2 *_3 d\left(\frac{\tilde{Z}+\frac{1}{2}}{x^2}\right)\right]\wedge d\Omega_3 + dual, \\ \tilde{h}^{-2} &= 2x\cosh H, \quad (dV)_{z\bar{z}} = \frac{i}{2x}\partial_x\tilde{Z}, \quad x\partial_x V = i(dz\partial_z - d\bar{z}\partial_{\bar{z}})\tilde{Z}, \quad \tilde{Z} = \frac{1}{2}\tanh H, \end{aligned} \tag{6.22}$$

This construction gives a supersymmetric solution of IIB supergravity as long as function $\tilde{Z}$ satisfies a linear differential equation:

$$4\partial_z\partial_{\bar{z}}\tilde{Z} + x\partial_x\left(\frac{\partial_x\tilde{Z}}{x}\right) = 0. \tag{6.23}$$

Moreover, as shown in [20], the system (6.22) describes a smooth geometry if and only if function $\tilde{Z}$ obeys some special Dirichlet boundary conditions in the plane $x = 0$:

$$x = 0: \qquad \tilde{Z} = \frac{1}{2} \text{ or } \tilde{Z} = -\frac{1}{2}. \tag{6.24}$$

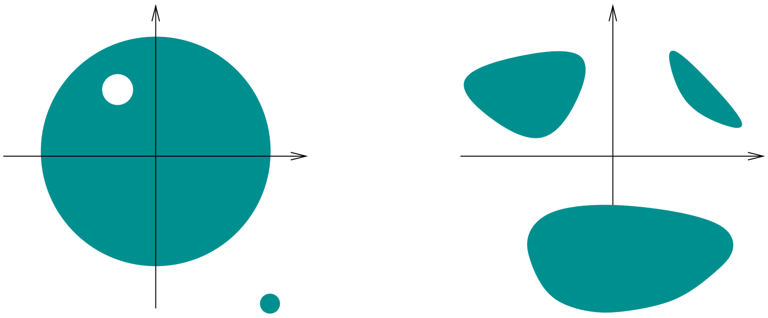

Figure 3: Boundary conditions for the 1/2-BPS geometries of [20]: (a) giant graviton and a dual giant, (b) generic distribution of droplets.

Thus the entire plane is divided into two types of regions and a typical boundary condition is depicted in figure 3b.

To compare these 1/2-BPS geometries with (6.2), we identify the three dimensional sphere appearing in (6.2) with S^3 in the metric (6.22), while embedding the Killing direction ψ from (6.2) into $\tilde{S}^3$:

$$d\tilde{\Omega}_3^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\tilde{\phi}^2. \tag{6.25}$$

Such identification follows from the field theory analysis. Our goal is to describe states in SYM on $R \times S^3$ which are constructed from elementary blocks (6.1). To preserve supersymmetry, one should look only at zero modes on S^3, this leads to the direct embedding of $d\Omega_3^2$ into the ansatz (6.2). The second sphere in (6.22) came from the SO(4) R-symmetry preserved by $\mathrm{Tr}X^n$, and a generic state constructed out of (6.1) breaks this isometry to U(1). To identify the embedding of this U(1) into SO(4), one should notice that the later group can be viewed as a set of rotations in four directions, while U(1) corresponds to rotations in one plane. This immediately leads to (6.25), in particular, it is clear that ψ coordinate, which corresponds to the U(1) translations, should be orthogonal to other directions. Notice that the 1/2-BPS geometries (6.22) can also be embedded into different class of 1/4-BPS solutions constructed in [32] by treating $\tilde{\phi} + \psi$ rather than ψ as a Killing vector preserved by the 1/4-BPS geometries. The corresponding embedding was discussed in [32, 23], but it appears to be irrelevant for viewing states $\prod_i \mathrm{Tr}X^{n_i}$ as a subset of objects constructed out of (6.1).

To summarize, we have argued that to embed the geometry (6.22) into the ansatz (6.2), one needs to equate $d\Omega_3^2$ appearing in both expressions and identify ψ direction of (6.2) with corresponding term in (6.25). Also the time coordinates in (6.2) and (6.22) should be the same, but it turns out the coordinate $\tilde{\phi}$ appearing in (6.22) does not belong to the subspace spanned by y and Kahler metric, while the shifted variable $\phi = \tilde{\phi} + t$ does.[20]

[20] The simplest way to see this is to compare the coordinate dependence of the Killing spinors in 1/2

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Starting with this identification, one can construct the complete map between 1/2-BPS and 1/4-BPS variables. The technical details are presented in the appendix B.1, and here we just summarize the results. First of all, it turns out that, to perform an embedding, one needs to rewrite (6.22) in terms of a new function D which is defined by the relations:

$$x\partial_x D = \frac{1}{2} - \tilde{Z}, \qquad V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})D. \tag{6.26}$$

The linear equation for $\tilde{Z}$ implies that D must be harmonic:

$$4\partial_z\partial_{\bar{z}}D + x^{-1}\partial_x(x\partial_x D) = 0. \tag{6.27}$$

The coordinates z_a and y appearing in (6.2) can be expressed through their counterparts from (6.22):

$$y = x\cos\theta, \quad z_1 = z, \quad z_2 = x\sin\theta e^{-D+i\phi}. \tag{6.28}$$

The 1/2-BPS metrics are governed by $\tilde{Z}$, the 1/4-BPS ones are parameterized by the Kahler potential K, and the relation (B.16):

$$y^{-1}\partial_y K = 2D - \log y \tag{6.29}$$

maps one description into another. Equations (6.26), (6.28) (6.29) provide local embedding of (6.22) into (6.2), and now we will relate the boundary conditions in these two cases.

We begin with observing that for the 1/2-BPS states the boundary conditions (6.24) can be reformulated in terms of D:

$$x = 0: \qquad \partial_x D = 0 \quad \text{or} \quad \partial_x D = \frac{1}{x} + O(x^0). \tag{6.30}$$

The first relation comes from the regularity condition ($\tilde{Z} = \pm\frac{1}{2} + O(x^2)$) for the solutions of (6.22). Notice that the boundary conditions (6.30) are identical to ones found for eleven-dimensional bubbling solutions in [20], and this analogy will be further explored in section 8. To interpret (6.30) in terms of the variables appropriate for the 1/4-BPS case, we need to rewrite the boundary conditions (6.30) in terms of y-derivatives. Using relations (B.19), we find:

$$y = 0: \qquad \begin{array}{ll} \cos\theta = 0: & \partial_y D = 0, \\ \cos\theta \neq 0: & \partial_y D = 0 \quad \text{or} \quad \partial_y D = \frac{1}{y} + O(y^0). \end{array} \tag{6.31}$$

Using (6.29) and (6.2), one can see that this translates into the correct boundary conditions[21] $Z = \pm\frac{1}{2}$, and now we will try to extract the shapes of the 1/4-BPS droplets. As in the 1/2-BPS case, the $y = 0$ hyperplane is divided into regions with $\partial_y D = 0$ and

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and 1/4-BPS cases, but one can also use a purely bosonic argument based on matching g_{tt} in (6.2) (see appendix B.1).

[21] As we will see in section 6.5, regularity also leads to an additional restriction on Kahler potential. This requirement is satisfied by the 1/2-BPS geometries, but we will not present the proof.

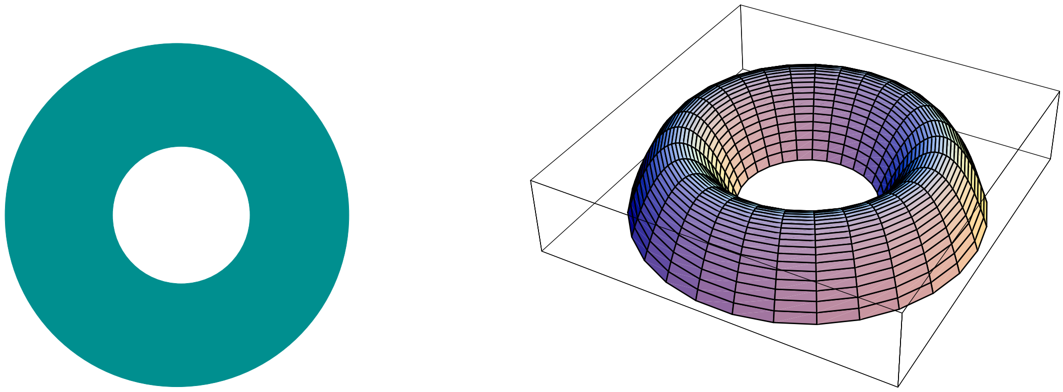

(a) (b)

Figure 4: Correspondence between boundary conditions in 1/2-BPS (a) and 1/4-BPS (b) cases.

$\partial_y D = \frac{1}{y}$, and pictorially one can use two different colors to distinguish between them. In the regions with $\partial_y D = \frac{1}{y} + O(y^0)$ one has the following relations:

$$x = 0, \qquad D = \log x + \hat{D}(z, \bar{z}) + O(x), \tag{6.32}$$

which allow us to rewrite (6.28) as

$$y = 0, \quad z_1 = z, \quad z_2 = \sin\theta e^{-\hat{D}+i\phi}. \tag{6.33}$$

This implies that the bubbles with $\partial_y D = \frac{1}{y}$ are defined by inequalities involving $|z_2|$ and z_1:

$$\partial_y D = \frac{1}{y}: \qquad |z_2| \le e^{-\hat{D}(z,\bar{z})}. \tag{6.34}$$

In particular, the coloring of the $y = 0$ hyperplane is invariant under phase shift in z_2, and an example is presented in figure 4b.

To summarize, we demonstrated that 1/2-BPS bubbling solutions of [20] can be embedded in the more general ansatz (6.2), and the map between two descriptions is given by (6.28), (6.29). Moreover, the 1/2-BPS geometries correspond to very simple boundary conditions in $y = 0$ plane: the walls between regions with $\partial_y D = 0$ and $\partial_y D = \frac{1}{y}$ are determined by the equation

$$z_2\bar{z}_2 = e^{-2\hat{D}(z_1,\bar{z}_1)}, \tag{6.35}$$

where $\hat{D} = D - \log x$ is a finite part of the harmonic function in the $\tilde{Z} = -\frac{1}{2}$ region.

6.3 Boundary conditions I: D branes

Let us now go back to the solution (6.2) and discuss supersymmetric D3 branes in this geometry. It turns out that there are two types of branes preserving eight supercharges, and, just as in the case of the D3-webs discussed in section 2, their profiles cannot be arbitrary. In this subsection we will show that a consistency of IIB supergravity leads to some restrictions on the locations of D3 branes. In the next subsection we will demonstrate that the allowed profiles are in a perfect agreement with results of the probe analysis. The SO(4) symmetry required by the SUSY algebra leads to two distinguished classes of branes, and we discuss them one-by-one. Using terminology of [30], the first class can be called ”giant gravitons”, while the second type represents ”dual giants”.

1. *Giant gravitons and holomorphic surfaces.*

   Let us consider branes which do not wrap the three-sphere. To preserve the SO(4) symmetry, these branes must be located at points where the radius of S^3 goes to zero, then, to avoid singularities in $g_{\psi\psi}$, y must vanish at the locations of the branes. Looking at the structure of F_5, we conclude that, in addition to being extended in t, the brane wraps ψ-direction. We can introduce two additional coordinates (w_1, w_2) on the D3 worldvolume, and they should be functions of $(z_a, \bar{z}_a)$. Following notation of section 4.1, we use (v_1, v_2) to parameterize the complement of (w_1, w_2) in the four-dimensional space spanned by $(z_a, \bar{z}_a)$. In the vicinity of the D3 brane the leading contribution to the radius of S^3 should not depend on the longitudinal coordinates, then definition of Z implies a decomposition (4.8) in the Kahler potential. Then, repeating the same arguments that led to (4.9), we can write the Kahler potential as a sum of two terms

   $$K = K_1(w_1, w_2) + K_2(v_1, v_2, \mathbf{x}) \tag{6.36}$$

   with divergent K_2 and finite K_1. Substituting this expansion into the Monge-Ampere equation appearing in (6.2), and matching finite w-dependent terms, we find a relation

   $$\partial_1\bar{\partial}_1 K_1 \ \partial_2\bar{\partial}_2 K_1 - \partial_1\bar{\partial}_2 K_1 \ \partial_2\bar{\partial}_1 K_1 = 0, \tag{6.37}$$

   which have been encountered before (see (4.10)). As was shown in section 4.1, this relation implies that $w = w_1 + iw_2$ is a holomorphic function of z_1, z_2. Then one can reparameterize the space transverse to the brane, so that $v = v_1 + iv_2$ is also holomorphic. We conclude that consistency of IIB supergravity requires the D3 branes to have a holomorphic profile in $(z_a, \bar{z}_a)$ directions. This conclusion agrees with the results of the probe analysis presented in the next subsection. It is also consistent with the fact that the D3 webs are described by holomorphic curves (see section 2), since asymptotically-flat webs can be obtained from the bubbling solutions by the procedure outlined in section 6.8.

2. *Dual giants.*

   Let us now consider the D3 branes which wrap S^3. To have a Lorentzian worldvolume, the brane should also be stretched along time direction, then six coordinates $(z_a, \bar{z}_a, y, \psi)$ can be used to parameterize the directions transverse to the brane. In particular, to preserve U(1) symmetry, the metric component $g_{\psi\psi}$ should go to zero at the location of the brane, then, to avoid a divergence in the warp factor of the sphere, we must require that $y = 0$. Thus we conclude that the symmetries of the problem require the ”dual giant gravitons” to be located at the points

   $$y = 0, \quad z_a = z_a^{(0)}. \tag{6.38}$$

To summarize, we demonstrated that, to have a consistent supergravity solution with eight supercharges, one should only allow two types of D-brane sources: they should either follow holomorphic profiles in the Kahler space while wrapping t and ψ directions, or they should wrap S^3 and t while being a point in the $(z_a, \bar{z}_a)$ subspace. Both types of branes must be located at $y = 0$, but giant gravitons sweep holomorphic surfaces in the subspace where $Z = -\frac{1}{2}$, while dial giants are localized at points in the regions where $Z = \frac{1}{2}$.

In section 5 it was demonstrated that in asymptotically-flat case a consistency of supergravity and a probe analysis lead to the same restrictions on the location of sources, this was a manifestation of the open/closed string duality. In the next subsection we will show that a similar agreement occurs for the giant gravitons.

6.4 Comparison to probe analysis

In this subsection we will analyze supersymmetric D-branes in the geometry (6.2). To mimic the discussion of asymptotically flat space presented in section 2, one should begin with studying branes in $AdS_5 \times S^5$ (since this is the closest analog of branes in flat space). Then the DBI action implies that, in a perfect agreement with results of the previous subsection, D3 branes must follow holomorphic profiles [24]. We will demonstrate that holomorphicity discovered in [24] must be formulated in terms of the complex coordinates z_a which were used in (6.2). We will also show that brane profiles must be holomorphic even in a general 1/4-BPS metric (6.2). The last part of this subsection will be devoted to the "dual giants", i.e. to branes wrapping S^3 in (6.2).

As reviewed in section 2, to identify supersymmetric branes in an arbitrary background, one needs to solve the kappa-symmetry projections (2.12). Presently we are interested in D3 branes in $AdS_5 \times S^5$ and we begin with analyzing the "original giant gravitons", which appear as pointlike objects on AdS space [30]. This implies that the brane is located at $\rho = 0$, then, recalling the embedding (6.4), one concludes that $y = 0$ while $r < 1$. Kahler potential (6.5) can be expanded in the vicinity of such points:

$$\Psi = 1 + \frac{y^2}{1-r^2} + O(y^4), \quad K = \frac{1}{2}\left[1 - y^2 \log(1-r^2) + y^2 \log y\right] + O(y^4) \qquad (6.39)$$

To evaluate $Z + \frac{1}{2}$, one can look at subleading terms K and use (6.2), but equation (6.6) gives a more direct route to the answer:

$$Z + \frac{1}{2} = h^2 \sinh^2 \rho = \frac{y^2}{\sin^4 \theta} + O(y^4) = \frac{y^2}{(1-r^2)^2} + O(y^4), \quad e^G = \frac{y}{1-r^2} + O(y^2)$$

Using this data, we can write the leading contributions to the metric appearing in (6.2):

$$\begin{aligned} ds_{10}^2 &= (1-r^2)\left[-(dt+\omega)^2 + 2\partial_a \bar{\partial}_b \tilde{K} dz^a d\bar{z}^b + d\psi^2\right] + \frac{1}{1-r^2}\left[dy^2 + y^2 d\Omega_3^2\right] \\ \omega &= i\left[\partial - \bar{\partial}\right]\tilde{K}, \quad \tilde{K} = -\frac{1}{2}\log(1-r^2), \quad r^2 = z_a \bar{z}_a. \end{aligned} \qquad (6.40)$$

We already know that D3 branes are located at $y = 0$, let us now specify the other coordinates of these objects. Since 1/4-BPS giant gravitons are expected to preserve the

U(1) part of the R-symmetry group, they should wrap ψ-coordinate, moreover, a position of the giant in the remaining directions should not depend ψ. The brane worldvolume extends in the time direction as well, but, since giant graviton is a rotating object, one is tempted to allow for time dependence of the transverse coordinates. Indeed, giant graviton moving in the metric (6.3) follows a trajectory with nontrivial $\phi_1(t)$, $\phi_2(t)$, but time dependence cancels out in the complex coordinates (6.4). Similar situation was encountered in the case of 1/2-BPS case [20], where giant gravitons turned out to be static in objects $y = 0$ plane. In the present case, z-coordinates of the D3-brane must be time-independent. Thus, to analyze the DBI projection (2.12), we can impose a static gauge:

$$t = \xi^0, \quad z_a = z_a(\xi^1, \xi^2), \quad \psi = \xi^3. \tag{6.41}$$

This choice leads to the following induced metric:

$$\begin{aligned} ds^2_{\text{ind}} &= (1-r^2)\left[-(d\xi^0 + i\partial_a\tilde{K}Dz^a - i\bar{\partial}_a\tilde{K}D\bar{z}^a)^2 + 2\partial_a\bar{\partial}_b\tilde{K}Dz^aD\bar{z}^b + d\xi_3^2\right], \\ Df &= \partial_{\xi^1}f d\xi^1 + \partial_{\xi^2}f d\xi^2, \end{aligned} \tag{6.42}$$

which allows to compute $\mathcal{L}$ introduced in (2.12):

$$\mathcal{L} = (1-r^2)^2 \det(2\partial_a\bar{\partial}_b\tilde{K}\partial_m z^a\partial_n\bar{z}^b) \tag{6.43}$$

The relations (6.40), (6.42), (6.43) can be easily generalized to a case of an arbitrary 1/4-BPS geometry (6.2):

$$\begin{aligned} ds^2_{10} &= h^{-2}\left[-(dt+\omega)^2 + 2\partial_a\bar{\partial}_b\tilde{K}dz^ad\bar{z}^b + d\psi^2\right] + h^2\left[dy^2 + y^2 d\Omega_3^2\right], \\ ds^2_{\text{ind}} &= h^{-2}\left[-(d\xi^0 + i\partial_a\tilde{K}Dz^a - i\bar{\partial}_a\tilde{K}D\bar{z}^a)^2 + 2\partial_a\bar{\partial}_b\tilde{K}Dz^aD\bar{z}^b + d\xi_3^2\right], \\ \mathcal{L} &= h^{-4}\det(2\partial_a\bar{\partial}_b\tilde{K}\partial_m z^a\partial_n\bar{z}^b), \end{aligned} \tag{6.44}$$

and from now on our discussion would refer to this general case. Using an intuition from $AdS_5 \times S^5$ solution, we will impose the static gauge (6.41).

To evaluate gamma matrices appearing in (2.12), we need expressions for some components of the (reduced) veilbein corresponding the ten dimensional metric (6.44):

$$e^{\mathbf{t}} = (dt+\omega), \quad e^{\psi} = d\psi, \quad e^{\mathbf{a}}, \quad e^{\bar{\mathbf{a}}}: \quad \delta_{ab}e^{\mathbf{a}}e^{\bar{\mathbf{b}}} = 2\partial_a\bar{\partial}_b\tilde{K}dz^ad\bar{z}^b \tag{6.45}$$

A nontrivial restriction on shape of branes comes from the requirement that the projector (2.12) does not break any of the 8 supercharges which are preserved by (6.2),[22] so we need to recall the structure of the relevant Killing spinors. While these spinors have not been explicitly written down in the literature, some useful information can be extracted

[22]Of course, the $AdS_5 \times S^5$ metric has additional supersymmetries which appear accidental from the point of view of 1/4-BPS analysis. While giant graviton is allowed break these "extra" supercharges, to be at least 1/4-BPS, it must preserve the ones which were explicit in (6.2).

from the relations between bilinears found in [19]:[23]

$$\bar\epsilon\Gamma_{\mathbf{ab}}\gamma_7\hat\sigma_1\epsilon=\bar\epsilon\Gamma_{\bar{\mathbf{a}}\bar{\mathbf{b}}}\gamma_7\hat\sigma_1\epsilon=0,\qquad \bar\epsilon\Gamma_{\mathbf{a}\bar{\mathbf{b}}}\gamma_7\hat\sigma_1\epsilon=\frac{1}{2}\delta_{\mathbf{ab}}\bar\epsilon\gamma_7\hat\sigma_1\epsilon \tag{6.46}$$

$$\bar\epsilon\gamma_7\hat\sigma_1\epsilon=\sqrt{y}e^{-G/2}h\epsilon^\dagger\epsilon,\qquad i\bar\epsilon\hat\sigma_1\epsilon=\sqrt{y}e^{G/2}h\epsilon^\dagger\epsilon \tag{6.47}$$

It is convenient to rewrite the last two relations in terms of a rotated spinor $\tilde\epsilon$:

$$\begin{aligned}\epsilon=e^{i\delta\gamma_7}\tilde\epsilon:\quad \bar\epsilon\gamma_7\hat\sigma_1\epsilon&=\cos 2\delta\ \tilde\epsilon^\dagger\Gamma^0\gamma_7\hat\sigma_1\tilde\epsilon+i\sin 2\delta\tilde\epsilon^\dagger\Gamma^0\hat\sigma_1\tilde\epsilon=\frac{e^{-G/2}}{\sqrt{e^G+e^{-G}}}\tilde\epsilon^\dagger\tilde\epsilon,\\ i\bar\epsilon\hat\sigma_1\epsilon&=-\sin 2\delta\ \tilde\epsilon^\dagger\Gamma^0\gamma_7\hat\sigma_1\tilde\epsilon+i\cos 2\delta\tilde\epsilon^\dagger\Gamma^0\hat\sigma_1\tilde\epsilon=\frac{e^{G/2}}{\sqrt{e^G+e^{-G}}}\tilde\epsilon^\dagger\tilde\epsilon.\end{aligned} \tag{6.48}$$

By setting

$$\sin 2\delta=\frac{e^{-G/2}}{\sqrt{e^G+e^{-G}}},\quad \cos 2\delta=\frac{e^{G/2}}{\sqrt{e^G+e^{-G}}}, \tag{6.49}$$

one can reformulate equations (6.48) as a projector

$$\Gamma^0\hat\sigma_1\tilde\epsilon=-i\tilde\epsilon. \tag{6.50}$$

Rewriting (6.46), (6.47) in terms of $\tilde\epsilon$, we find very simple relations:

$$\tilde\epsilon^\dagger\Gamma_{\mathbf{ab}}\tilde\epsilon=\tilde\epsilon^\dagger\Gamma_{\bar{\mathbf{a}}\bar{\mathbf{b}}}\tilde\epsilon=0,\quad \tilde\epsilon^\dagger\Gamma_{\mathbf{a}\bar{\mathbf{b}}}\tilde\epsilon=\frac{1}{2}\delta_{\mathbf{ab}}\tilde\epsilon^\dagger\tilde\epsilon, \tag{6.51}$$

which imply that $\tilde\epsilon$ is annihilated by the holomorphic gamma matrices ($\Gamma_{\mathbf{a}}\tilde\epsilon=0$). Recalling (6.48), we observe that the same relation is satisfied by the original spinor ϵ:

$$\Gamma_{\mathbf{a}}\epsilon=0. \tag{6.52}$$

After this brief review of the Killing spinor on the geometry (6.2), we are ready to analyze the DBI projector (2.12):

$$\Gamma=i\mathcal{L}^{-1}(1-r^2)^2\sigma_2\otimes\Gamma_{t\psi}\frac{\partial X^m}{\partial\xi^1}\frac{\partial X^n}{\partial\xi^2}\tilde\gamma_{mn},\quad \Gamma\epsilon=\epsilon \tag{6.53}$$

Here indices m,n go from one to four and coordinates X^m cover the subspace of (6.40) spanned by $z^a,\bar z^a$, while the action $\mathcal{L}$ is computed using the metric (6.44). The matrices $\tilde\gamma_{mn}$ are constructed using the veilbein $e^{\mathbf{a}}$, $e^{\bar{\mathbf{a}}}$ since contribution of $e^{\mathbf{t}}_m$ disappears from γ_{tm}:

$$\gamma_{tm}=e^{\mathbf{t}}_t e^{\mathbf{t}}_m\Gamma_{\mathbf{tt}}+e^{\mathbf{t}}_t e^{\mathbf{m}}_m\Gamma_{\mathbf{tm}}=\gamma_t\tilde\gamma_m. \tag{6.54}$$

Applying $\Gamma_{\mathbf{ab}}$ to the projector (6.53) and using (6.52), we arrive at a relation

$$\frac{\partial\bar z_1}{\partial\xi^1}\frac{\partial\bar z_2}{\partial\xi^2}-\frac{\partial\bar z_2}{\partial\xi^1}\frac{\partial\bar z_1}{\partial\xi^2}=0, \tag{6.55}$$

[23] The relations (6.46) were derived in [19] for a complex spinor in IIB supergravity. Translation to the alternative conventions used in (2.12), is performed in (6.52). Notice that relations (6.46), (6.47) have extra factors of $\hat\sigma_1$ in comparison with [19], these insertions arise from rewriting the reduced spinor of [19] as a complex spinor in ten dimension (see [19] for the definition of gamma matrices).

which implies a functional dependence $\bar{z}_2 = \bar{f}(\bar{z}_1)$, $z_2 = f(z_1)$. Thus, by studying the DBI projection (2.12), we showed that the the profiles of supersymmetric D3 branes in the geometry (6.2) must be holomorphic:

$$f(z_1, z_2) = 0. \tag{6.56}$$

This outcome of the open string analysis is in a perfect agreement with closed string picture which was discussed in section 6.3: there it was shown that only holomorphic sources are consistent with equations of motion of IIB supergravity.

In the case of $AdS_5 \times S^5$ it is interesting to compare our holomorphic curves with analysis of the giant gravitons presented in [24]. Mikhailov proposed to "fill in" the five-dimensional sphere by introducing an fictitious radial coordinate R and writing a six-dimensional metric of $\mathbf{C}^3$ as

$$ds_6^2 = dR^2 + R^2 d\Omega_5^2 = dw_a d\bar{w}_a. \tag{6.57}$$

Then it was shown that supersymmetric D3 branes must be located at intersections of two-dimensional holomorphic surfaces in $\mathbf{C}^3$ (with some additional time dependence) and a sphere $R = 1$. A generic surface $f(w_1 e^{it}, w_2 e^{it}, w_3 e^{it}) = 0$ gives rise to a giant graviton preserving four supercharges, while a surface $f(w_1 e^{it}, w_2 e^{it}) = 0$ leads to a 1/4-BPS object. Matching this with equation (6.56), we find a map between w_a and z_a coordinates ($z_a = w_a e^{it}$), then definitions (6.4) lead to relations between w_a and standard coordinates on $AdS_5 \times S^5$ (see (6.3)):

$$w_1 = \cosh\rho[\cos\theta\cos\alpha e^{i\phi_1}] = \mu_1\cosh\rho, \quad w_2 = \cosh\rho[\cos\theta\sin\alpha e^{i\phi_2}] = \mu_2\cosh\rho \tag{6.58}$$

Here μ_1 and μ_2 are two of the six coordinates defining the five-sphere:

$$\sum_{a=1}^{3} \bar{\mu}_a \mu_a = 1, \tag{6.59}$$

so we arrive at an identification $R = \cosh\rho$. This analysis demonstrates that, rather than being an artificial parameter, Mikhailov's radial coordinate has a very simple meaning for $AdS_5 \times S^5$. Moreover, by introducing coordinates z_a and demonstrating holomorphicity, we showed how to define a similar "radial direction" for the most general 1/4-BPS geometry as well: a holomorphic surface $f(z_a) = 0$ is naturally parameterized by one complex coordinate and y. The worldvolume of the D3 brane is obtained by taking an intersection of this surface with any region $\{y = 0, Z = \frac{1}{2}\}$, and fibering ψ and t over it. A pictorial representation of this construction is given in figure 5. Notice that, since ψ direction shrinks at the boundary of $\{y = 0, Z = \frac{1}{2}\}$ region, the worldvolume of the resulting D3 brane is a manifold without boundaries.

Let us now make a brief comment about the "dual giant graviton". This object wraps time direction and three-dimensional sphere which remains unbroken in the ansatz (6.2), so, by counting dimensions, one concludes that the dual giant should be located at a point $z_a = z_a^{(0)}$ on the $y = 0$ surface, and at that point $Z = -\frac{1}{2}$. This result trivially agrees with discussion presented in section 6.3.

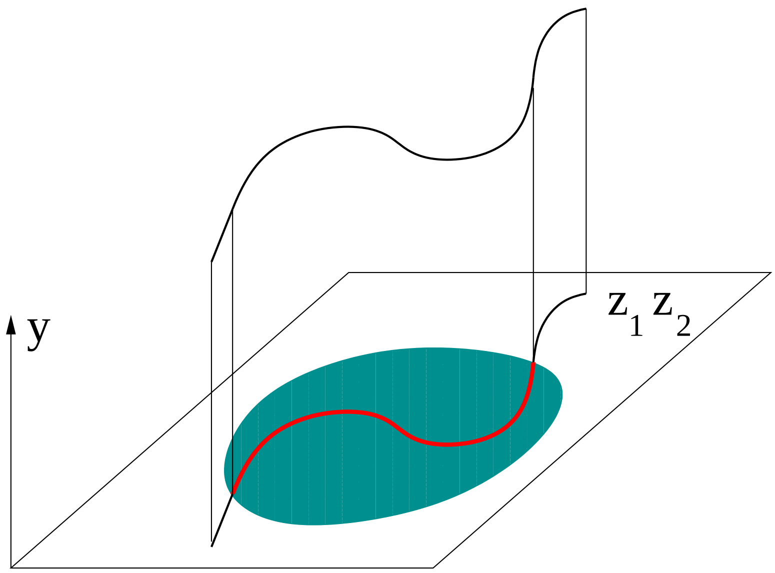


Figure 5: Giant gravitons and holomorphic surfaces. To construct a worldvolume of a D3 brane, one should take an intersection of the holomorphic surface with $\{y = 0, Z = \frac{1}{2}\}$ region (this intersection is shown in red), and fiber ψ and t over it. In the case of $AdS_5 \times S^5$, this picture gives a geometric interpretation of the radial coordinate introduced in [24]

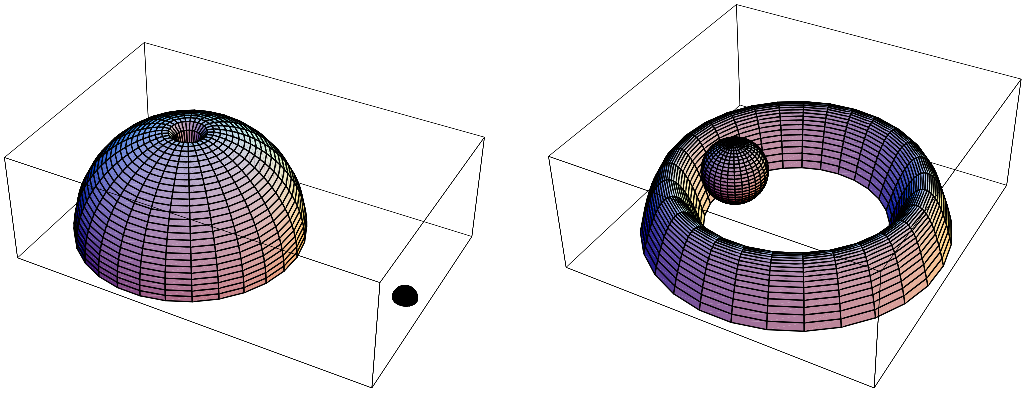
Let us summarize the results of this subsection. By analyzing supersymmetry conditions for D3 branes on a general geometry (6.2), we showed that there are two types of interesting objects: "giant graviton" which wraps t, ψ directions and follows a holomorphic profile $f(z_1, z_2) = 0$ in the $Z = -\frac{1}{2}$ subset of $y = 0$ space, and "dual giant" which wraps t, S^3 and occupies a point in $Z = \frac{1}{2}$, $y = 0$ subspace. No other object can preserve eight supercharges. These results of open string analysis agree perfectly with supergravity discussion presented in section 6.3.

6.5 Boundary conditions II: regular droplets

After reproducing the correct boundary conditions corresponding to probe D3 branes, we now study geometries produced by the stacks of the branes. While for the small number of branes the boundary conditions for (5.8) and (6.2) are similar, the results for multiple branes are very different. This phenomenon has already been seen in the case of 1/2-BPS solutions: for asymptotically-flat geometry one can simply superpose stacks of D3 branes and each element in the stack has exactly the same location. On the contrary, the branes in $AdS_5 \times S^5$ repel each other[24] and form non-compressible droplets [20]. This droplets change the topology of spacetime and, as a result of such bubbling, the geometry remains smooth everywhere. A similar phenomenon is expected to take place for the configurations with lower supersymmetry[25] and this subsection will be devoted to deriving the "bubbling picture" for the 1/4-BPS geometries.

[24]The field theory manifestation of this phenomenon was discussed in [29].

[25]See [31] for the relevant field theory analysis.



(a) (b)

Figure 6: Boundary conditions in the 1/4-BPS case: giant graviton & dual giant (a) and a generic distribution of droplets (b). To simplify the picture, we use vertical axis for $|z_2|$ and suppress the phase of z_2.

Let us begin with recalling the results pertaining to 1/2-BPS case [20]. The gravity solutions had $\mathrm{SO}(4) \times \mathrm{SO}(4)$ isometry and a coordinate y was defined as a product of the warp factors corresponding to the two three-spheres. At $y = 0$ one of the spheres had to collapse to zero size, and such contraction would lead to a singularity in the geometry unless some special boundary conditions were imposed. It turned out that the solutions were parameterized by one harmonic function z and regularity led to the requirement that $Z = \pm\frac{1}{2}$ in $y = 0$ plane [20]. Then the entire plane was separated into two types of regions (see figure 3): one of the three-spheres collapsed in the light region and another one did so in the dark region. There were no restrictions on the curves separating the regions. This arbitrariness was in a complete agreement with brane probe analysis: the (dual) giant gravitons corresponded to light (dark) points (see figure 3a) which could be combined to give droplets with arbitrary shapes. As we will see in a moment, in the 1/4-BPS case the situation is completely different.

In a complete analogy with 1/2-BPS case, we observe that y-coordinate in (6.2) is a product of the two warp-factors, so on $y = 0$ hypersurface either S^3 or S^1 collapses to zero size. Thus, the entire hypersurface is again divided into two types of regions (some examples are presented in figure 6). We will begin with demonstrating that in the interior of each droplet the geometry remains regular, once certain restrictions on a Kahler potential are imposed. It turns out that, unlike 1/2-BPS droplets which could have arbitrary shapes, their 1/4-BPS counterparts lead to singular solutions unless some additional condition is satisfied by their "walls". One can suspect that this should be the case by observing that the 1/4-BPS probe branes must follow holomorphic profiles. If arbitrary shapes of the droplets were allowed, one could always take a degenerate limit leading to a source wrapping a non-holomorphic surface. This would imply an existence of some exotic D3 brane which is not allowed in string theory. Fortunately, as we will show below, this situation is ruled out by the regularity conditions for the geometries, which imposes certain requirements on the boundaries of the droplets. But before we start analyzing the boundaries, let us demonstrate the regularity at the interior points.

Droplets and regularity conditions. As already mentioned, one of the spheres collapses at $y = 0$ surface, so one needs to check that the metric remains regular there.

We will begin with analyzing a vicinity of a point where y goes to zero, while $g \equiv ye^{-G}$ remains finite. The definition of Z leads to very simple leading terms in h and in the Kahler potential:[26]

$$Z = -\frac{1}{2} + \frac{y^2}{g^2}, \quad h^{-2} = g \quad K = \int dy\ y \log y + K_0(z,\bar{z}) + y^2 K_1(z,\bar{z}) + O(y^4). \quad (6.60)$$

Then Monge-Ampere equation implies that

$$\det h_{a\bar{b}} = y^4 g^{-2} \exp[K_1(z,\bar{z})] + O(y^6). \quad (6.61)$$

We can now write the leading contribution to the metric

$$\begin{aligned} ds^2_{10} &= g\left[-(dt+\omega)^2 + d\psi^2\right] + g^{-1}\left[2g^2 y^{-2} h_{a\bar{b}} dz^a d\bar{z}^b + dy^2 + y^2 d\Omega_3^2\right], \quad (6.62)\\ \omega &= i(\bar{\partial} - \partial)K_1, \end{aligned}$$

and, to demonstrate regularity, one needs to show that *all* components of $h_{a\bar{b}}$ vanish as y^2 (the restriction (6.61) on the determinant is not sufficient[27]). The leading contribution to (6.61) leads to an equation for K_0:

$$\partial_1\bar{\partial}_1 K_0 \partial_2\bar{\partial}_2 K_0 - \partial_1\bar{\partial}_2 K_0 \partial_2\bar{\partial}_1 K_0 = 0, \quad (6.63)$$

then analysis of section 4 implies that $K_0 = K_0(w,\bar{w})$, where w is a holomorphic function of z_a. Unfortunately the Monge-Ampere equation does not impose further restrictions on function K_0, so, to ensure regularity, one should supplement the condition $Z = -\frac{1}{2}$ by the requirement that $K_0 = 0$:

$$y = 0: \ \ Z = -\frac{1}{2}, \quad \partial_a\bar{\partial}_b K(z,\bar{z}, y=0) = 0. \quad (6.64)$$

Notice that the last condition was missed in [23], but it is crucial for enforcing regularity. Moreover, as we will show in section 6.6, both relations in (6.64) are needed to specify the solution uniquely. As expected, these relations are satisfied by the Kahler potential (6.5) is $r < 1$ and by the potential (6.14), (6.20) if $r > 1$. Using the maximum principle for equation (6.63) and fixing the gauge freedom in K, one can rewrite the last relation in (6.64) as two requirements: $K_0 = K(z,\bar{z}, y = 0)$ must regular in the region where $Z = -\frac{1}{2}$, and K_0 should vanish on the boundaries of this region:

$$y = 0: \ \ Z = -\frac{1}{2}, \quad K_0(z,\bar{z}) - \text{regular}, \quad K_0(z,\bar{z})|_{\text{bndry}} = 0. \quad (6.65)$$

[26]Notice that the Monge-Ampere equation appearing in (6.2) can be rewritten in terms of a new variable y^2, an absence of branch cuts in this new description implies that K has expansions in integer powers of y^2.

[27]This was a loophole in the proof of regularity presented in the appendix D of [23].

While this condition looks weaker than (6.64), equation (6.63) makes these two relations equivalent, and each of them is stronger than the requirement $Z = -\frac{1}{2}$. As AdS example (6.5) shows, strictly speaking function K_0 is not well-defined on the wall between different regions, so, while (6.65) provides a good heuristic picture for the boundary conditions, the relation (6.64) is more rigorous, and it will be used in the remaining part of the paper.

Let us now consider a vicinity of a point where y goes to zero, while $g \equiv ye^G$ remains finite: at such point the warp factor in front of S^3 remains finite, while the ψ-circle collapses. We again find an asymptotic expansion of the Kahler potential:

$$Z = \frac{1}{2} + \frac{y^2}{g^2}, \quad h^{-2} = g \quad K = -\int dy\ y \log y + K_0(z, \bar{z}) + y^2 K_1(z, \bar{z}) + O(y^4), \tag{6.66}$$

but now it is sufficient to keep only K_0 since the metric becomes

$$ds_{10}^2 = g\left[-dt^2 + d\Omega_3^2\right] + g^{-1}\left[2\partial_a\bar{\partial}_b K_0 dz^a d\bar{z}^b + dy^2 + y^2 d\psi^2\right] \tag{6.67}$$

Clearly this metric describes a regular space, so we arrive at a complement of (6.64)

$$y = 0: \ \ Z = \frac{1}{2}. \tag{6.68}$$

To summarize, we saw that the hypersurface $y = 0$ splits into two types of regions: S^3 collapses to zero size when $Z = -\frac{1}{2}$ and ψ-circle shrinks when $Z = \frac{1}{2}$. We also demonstrated that, in a complete analogy with [20], the metric remains regular in the interior points of the regions (although, there is an additional requirement: $\partial_a\bar{\partial}_b K = 0$ inside droplets with $Z = -\frac{1}{2}$). However, in contrast to the 1/2-BPS case where regions were allowed to have arbitrary shapes [20], in the present situation additional regularity conditions arise on the boundary of the droplets (where both S^3 and S^1 have vanishing warp-factors). Let us discuss the relevant restrictions.

Boundaries of the droplets: an example. To analyze regularity conditions for the wall separating different regions, it is convenient to begin with example which shows how S^3 and S^1 are incorporated in a patch of flat space. The simplest solution which fits in the ansatz (6.2) is $AdS_5 \times S^5$ and we already presented its local structure in section 6.2. Now let us discuss the appropriate boundary conditions.

The regions with $Z = \frac{1}{2}$ and $Z = -\frac{1}{2}$ are separated by the three sphere $r = 1$ (see equation (6.7)) and at each point on this sphere the space is locally flat. To see this more explicitly, we make an expansion of the metric in the vicinity of such point. Introducing new coordinates[28] v, R, ζ:

$$v = r^2 - 1, \quad R = \sqrt{(r^2 + y^2 - 1)^2 + 4y^2}, \quad \cos^2\zeta = \frac{ye^G}{R}, \tag{6.69}$$

we find the following relations:

$$ye^{-G} = R\sin^2\zeta, \quad y = \frac{R}{2}\sin 2\zeta, \quad Z = \frac{1}{2}\cos 2\zeta, \quad h^{-2} = R. \tag{6.70}$$

[28]Notice that for $AdS_5 \times S^5$ one has a relation $\sqrt{(r^2 + y^2 - 1)^2 + 4y^2} = \sin^2\theta + \sinh^2\rho = 2y\cosh G$, so ζ takes real values.

To evaluate the metric, we need two more ingredients: the geometry on the Kahler base and one-form ω. Since Kahler potential (6.5) depends on $z_a, \bar{z}_a$ only through v, the four dimensional metric can be written as

$$2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b = 2\partial_v K dz_a d\bar{z}_a + 2\partial_v^2 K|\bar{z}_a dz_a|^2, \quad v = z_a\bar{z}_a - 1. \tag{6.71}$$

The derivatives of (6.5) can be easily evaluated:

$$\begin{aligned}
\partial_v K &= \frac{1}{4(1+v)}\left[(v-y^2)+R\right] = \frac{R}{4}(\cos 2\zeta + 1) + O(R^2),\\
\partial_v^2 K &= \frac{1}{4}(\cos 2\zeta + 1) + O(R),
\end{aligned} \tag{6.72}$$

and used to compute the leading contribution to the metric on the base:

$$2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b = \left(Z+\frac{1}{2}\right)\left[R dz_a d\bar{z}_a + |\bar{z}_a dz_a|^2\right] + \dots \tag{6.73}$$

Next we simplify the expression for the one-form:

$$\begin{aligned}
\omega &= \frac{i}{2y}\left[\bar{\partial}\partial_y K - \partial\partial_y K\right] = \frac{1}{8(1+v)}\left(-2+\frac{4+v+y^2}{R}\right)\eta = \left[\frac{1}{2R}+O(1)\right]\eta,\\
\eta &\equiv i(z_a d\bar{z}_a - \bar{z}_a dz_a)
\end{aligned} \tag{6.74}$$

Using this data, we find the leading contribution to the ten-dimensional metric:

$$\begin{aligned}
ds_{10}^2 &\approx -R\left(dt+\frac{\eta}{2R}\right)^2 + \frac{1}{R}\left[R dx_\perp dx_\perp + \frac{1}{4}(dv^2+\eta^2) + dy^2\right] + R(c_\zeta^2 d\Omega_3^2 + s_\zeta^2 d\psi^2)\\
&\approx -dt\eta + dx_\perp dx_\perp + \frac{dR^2}{4R} + R(d\zeta^2 + \cos^2\zeta d\Omega_3^2 + \sin^2\zeta d\psi^2)
\end{aligned} \tag{6.75}$$

Since this is a metric of flat space, the solution is regular at the point $R=0$, and regularity cannot be affected by the subleading terms. Of course this result was expected since we were discussing $AdS_5 \times S^5$, but it was important to unravel the precise mechanism which makes the geometry regular, since we want to generalize it to other solutions.

Shapes of the droplets. Let us now use the lessons from $AdS_5 \times S^5$ to find the restrictions imposed by regularity. As already discussed, the necessary condition for (6.2) to describe a regular geometry is a decomposition of $y=0$ hypersurface into droplets with $Z=-\frac{1}{2}$. On the boundary separating two types of regions warp factors for both spheres should vanish, so it is convenient to define spherical coordinates by mimicking (6.70):

$$\tan\zeta = e^{-G}, \quad R = \frac{2y}{\sin 2\zeta}: \quad Z = \frac{1}{2}\cos 2\zeta, \quad h^{-2} = R. \tag{6.76}$$

Notice that the relations (6.69) were specific to $AdS_5 \times S^5$ case and they will not hold for a general solution. Since coordinate R measures a distance from the wall, we will be interested in the leading terms in R-expansion.

Regularity condition requires that the leading order of $v \equiv R\cos 2\zeta$ does not depend on y-coordinates, but rather it is a function on the Kahler base. To see this, we rewrite metric in five-dimensional subspace spanned by y, Ω_3, ψ:

$$\begin{aligned} ds_5^2 &\equiv h^2 dy^2 + y(e^G d\Omega_3^2 + e^{-G} d\psi^2) \\ &= \sin^2 2\zeta \frac{dR^2}{4R} + R\cos^2 2\zeta d\zeta^2 + R(\cos^2\zeta d\Omega_3^2 + \sin^2\zeta d\psi^2) \end{aligned} \tag{6.77}$$

In the ten-dimensional space this metric is combined with contribution coming from the Kahler base, and, to describe regular geometry, the sum should give a metric of the flat space. In other words, in (R,ζ) subspace, the Kahler metric should contribute the difference between flat six-dimensional space and (6.77):

$$ds^2_{\text{flat}} - ds_5^2 = \frac{1}{R}(dv)^2. \tag{6.78}$$

Thus the one form dv must lie in the Kahler subspace, i.e. $\partial_y v = 0$ at least in the leading order in R. Since one can easily invert the relations between (R,ζ) and (v,y), we conclude that the leading contributions to (R,ζ) depend on the Kahler base only through one real function $v(z_a,\bar z_a)$. In the $AdS_5\times S^5$ case this statement was true globally, but for a general 1/4-BPS geometry it holds only in the vicinity of a point on the "wall".

To proceed we need some additional information about Kahler potential. It can be extracted from the definition of Z:

$$\cos 2\zeta = -y\partial_y(y^{-1}\partial_y K) \tag{6.79}$$

Expressing $\cos 2\zeta$ through v and y ($\cos 2\zeta = \frac{v}{\sqrt{v^2+4y^2}}$), one can easily integrate this equation:

$$K = \frac{v}{8}\sqrt{v^2+4y^2} + \frac{y^2}{2}\left[\log(v+\sqrt{v^2+4y^2}) - \log y\right] + K_0(z,\bar z) + y^2 K_1(z,\bar z). \tag{6.80}$$

Since various warp factors depend on the four-dimensional base only through v, it is clear that this coordinate parameterizes a direction transverse to the wall. We can also introduce three longitudinal coordinates, and in the leading order one expects to have translational invariance in those directions. This implies that the leading contribution to Kahler potential should be a function of v and y only. In particular, K_0 and K_1 appearing in (6.80) depend on their arguments only through v.

Recalling an expression for the one-form ω:

$$\omega = \frac{i}{2y}\partial_y\partial_v K(\partial-\bar\partial)v \equiv \frac{1}{y}\partial_y\partial_v K\eta \sim \frac{1}{R}\eta, \quad \eta \equiv \frac{i}{2}(\partial-\bar\partial)v, \tag{6.81}$$

we can evaluate the leading terms in the metric (6.2):

$$\begin{aligned} ds^2 &= -2R\omega dt - R\omega^2 + \frac{1}{Rc_\zeta^2}\left[2\partial_v^2 K|\partial v|^2 + 2\partial_v K\partial\bar\partial v + c_\zeta^2 dy^2\right] + R(c_\zeta^2 d\Omega_3^2 + s_\zeta^2 d\psi^2) \\ &= -2R\omega dt - R\omega^2 + \frac{1}{R\cos^2\zeta}\left[2\partial_v^2 K|\partial v|^2 + 2\partial_v K\partial\bar\partial v - \frac{1}{4}\cos^2\zeta dv^2\right] \\ &\quad + \frac{dR^2}{4R} + R(d\zeta^2 + \cos^2\zeta d\Omega_3^2 + \sin^2\zeta d\psi^2) \end{aligned} \tag{6.82}$$

The last line gives a regular metric on R^6, so, to avoid singularity, the second line should parameterize $R^{1,3}$ in the vicinity of the wall. Let us analyze this four-dimensional metric in more detail:

$$\begin{aligned} ds_4 &= -2R\omega dt - R\omega^2 + \frac{1}{R\cos^2\zeta}\left[2\partial_v^2 K|\partial v|^2 + 2\partial_v K\partial\bar{\partial}v - \frac{1}{4}\cos^2\zeta dv^2\right] \qquad (6.83)\\ &= -2R\omega dt - \left(\frac{R}{y^2}(\partial_y\partial_v K)^2 - \frac{2\partial_v^2 K}{R\cos^2\zeta}\right)\eta^2 + \frac{2\partial_v K}{R\cos^2\zeta}\partial\bar{\partial}v + \frac{1}{4R}\left(\frac{2\partial_v^2 K}{\cos^2\zeta} - 1\right)dv^2 \end{aligned}$$

Since $\partial_v K \sim R$ in the vicinity of the wall,[29] the $\partial\bar{\partial}v$ term in the metric remains regular. The contributions proportional to dv^2 and η^2 are independent and naively they both look singular, so the divergences should cancel in both terms. This leads to the following relations:[30]

$$\partial_v^2 K = \frac{1}{2}\cos^2\zeta + O(R), \quad \partial_y\partial_v K = \frac{y}{R} + O(R), \qquad (6.84)$$

which completely determine the leading contributions to K.

Using this information, the four dimensional metric (6.83) can be rewritten in terms of two functions λ_1 and λ_2 which remain finite in the vicinity of the wall:

$$ds_4^2 = -2\eta dt + \partial\bar{\partial}v + \lambda_1\eta^2 + \lambda_2 dv^2 \qquad (6.85)$$

To avoid singularities in the ten-dimensional metric, the leading contributions to λ_1 and λ_2 should not depend on (y, v). Then regularity of the metric (6.85) requires an existence of a holomorphic one-form ξ, such that

$$\partial\bar{\partial}v + \lambda_1 dv^2 + \lambda_2\eta^2 = \xi\bar{\xi} + O(v). \qquad (6.86)$$

Indeed, to make the metric (6.85) regular, the matrix appearing in the left-hand side of the last equation should have rank two. This can only be accomplished if (anti)holomorphic terms $dz^a dz^b$, $d\bar{z}^a d\bar{z}^b$ cancel out in the lhs of (6.86), leading to holomorphicity of ξ ant to a relation $\lambda_2 = 4\lambda_1$. Moreover, on the surface $v = 0$, ξ describes a one-form in a two-dimensional space, so, by a change of coordinates, it can be always be written as $\xi = fdw$. An original definition of ξ implies that function w is holomorphic.

Now equation (6.86) can be rewritten as a relation between hermitean 2×2 matrices, which should be supplemented by requiring dv and dw to be independent:

$$\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v = g\partial_a w\bar{\partial}_b\bar{w} + O(v), \quad \det(\partial_a v\partial_b w)|_{v=0} \neq 0. \qquad (6.87)$$

We conclude that a droplet whose boundary is defined by equation $v(z_a, \bar{z}_a) = 0$ leads to a smooth metric if and only if function v satisfies (6.87). Notice that these relations are invariant under holomorphic reparametirizations which are regular on the $v = 0$ surface.

[29]This is obvious for first two terms in (6.80) and for $y^2 K_1$. To argue that $\partial_v K_0 \sim R$, we recall that Kahler potential has vanishing derivatives in $Z = -\frac{1}{2}$ region (see equation (6.64)), then continuity requires that $\partial_v K_0 \sim R$ on the boundary.

[30]One should be able to derive (6.84) by analyzing the Monge-Ampere equation in the vicinity of the wall, but we will not discuss this further.

To check the conditions (6.87), it is convenient to start with evaluating function λ. To do so, we compute the determinant on both sides of the first relation in (6.87):

$$\det(\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v) = O(v) \tag{6.88}$$

Moreover, since $\partial_a w$ and $\partial_a v$ are independent, equation appearing in (6.87) implies that

$$\det(\partial_a\bar{\partial}_b v)\big|_{v=0} \neq 0. \tag{6.89}$$

Once function λ is determined, one needs to check the remaining differential condition

$$\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v = \xi_a\bar{\xi}_b + O(v), \quad \xi_a dz^a = f\ dw + v\xi'. \tag{6.90}$$

The requirement (6.87) leads to rather nontrivial restrictions on the surfaces separating the droplets, and, to illustrate this fact, we consider few examples.

Examples of regular and singular droplets. The $AdS_5\times S^5$ example has already been discussed before, and, as a consistency check, we now demonstrate that conditions (6.88) and (6.89) are satisfied for that solution. Function v for this case was introduced in (6.69), so we find

$$\begin{aligned} v &= z_a\bar{z}_a - 1, \quad M_\lambda = \|\partial_a\bar{\partial}_b v + \lambda\partial_a v\bar{\partial}_b v\| = \begin{pmatrix} 1+\lambda z_1\bar{z}_1 & \lambda z_2\bar{z}_1 \\ \lambda z_1\bar{z}_2 & 1+\lambda z_2\bar{z}_2 \end{pmatrix}, \\ \det M_\lambda &= 1+\lambda z_a\bar{z}_a = 1+\lambda+\lambda v \end{aligned} \tag{6.91}$$

Thus the relation (6.88) is satisfied for $\lambda=-1$, and corresponding one-form is

$$\xi = z_2 dz_1 - z_1 dz_2 = z_1 z_2\ d\log\frac{z_1}{z_2}, \tag{6.92}$$

i.e. the requirement (6.90) is also satisfied. As expected, we found the the wall located at $z_a\bar{z}_a = 1$ leads to a regular solution.

Inspired by this example, one may consider the most general quadratic function of z_a and $\bar{z}_a$:

$$v = h_{ab}z_a\bar{z}_b + A_{ab}z_a z_b + \bar{A}_{ab}\bar{z}_a\bar{z}_b - B \tag{6.93}$$

It is clear that if $\det h = 0$, then (6.89) is violated, so such v would lead to a singular solution. Assuming that h_{ab} is a non-degenerate matrix, we can use linear transformations of z_a to diagonalize it:[31] $h'_{ab} = \delta_{ab}$. The residual U(2) invariance can be used to put v in one of the two canonical forms parameterized by real numbers a, b:

$$\begin{aligned} \mathrm{I}: \qquad v &= z_a\bar{z}_a + \left(az_1z_2 + \frac{b}{2}z_1^2 + cc\right) - B \\ \mathrm{II}: \qquad v &= z_1\bar{z}_1 - z_2\bar{z}_2 + \left(az_1z_2 + \frac{b}{2}z_1^2 + cc\right) - B \end{aligned} \tag{6.94}$$

[31] Since the criteria of regularity (6.87) are invariant under holomorphic reparameterizations, they are not affected by such transformations.

Equation (6.88) can be easily analyzed in each case, and, requiring function λ to remain finite at all points on the $v=0$ surface, we conclude that λ must be constant:

$$\begin{aligned}
\mathrm{I}: \qquad \det M_\lambda =& 1+\lambda\partial_a v\bar{\partial}_a v = 1+\lambda|\bar{z}_1+az_2+bz_1|^2+\lambda|\bar{z}_2+az_1|^2\\
=& 1+\lambda(1+a^2)(v+B)+\lambda b^2 z_1\bar{z}_1\\
&+\lambda\left[(1-a^2)\left(az_1z_2+\frac{b}{2}z_1^2\right)+abz_1\bar{z}_2+cc\right]\\
\det M_\lambda =& O(v): \quad b=0, \quad \lambda=-\frac{1}{B(1+a^2)}, \quad a(1-a^2)=0 \qquad (6.95)\\
\mathrm{II}: \qquad -\det M_\lambda =& 1+\lambda|\bar{z}_1+az_2+bz_1|^2-\lambda|\bar{z}_2-az_1|^2\\
=& 1+\lambda(1-a^2)(v+B)+\lambda b^2 z_1\bar{z}_1\\
&+\lambda\left[(1+a^2)\left(az_1z_2+\frac{b}{2}z_1^2\right)+abz_1\bar{z}_2+cc\right]\\
\det M_\lambda =& O(v): \quad b=0, \quad a=0, \quad \lambda=-\frac{1}{B}. \qquad (6.96)
\end{aligned}$$

The first case gives two possible values of a: $a=0$ reduces to the case of the sphere which was discussed before, while $a=1$ leads to a surface described by the equation

$$v=|z_1+\bar{z}_2|^2-B=Z\bar{Z}-B, \quad Z\equiv z_1+\bar{z}_2. \tag{6.97}$$

Let us check whether this function satisfies the relation (6.90). We begin with computing the one-form ξ and its differential:

$$\begin{aligned}
\partial\bar{\partial}v-\frac{1}{2B}|\partial v|^2 &= \frac{1}{2B}|\bar{Z}dz_1-Zdz_2|^2+O(v):\\
\xi &= \frac{1}{\sqrt{2B}}(\bar{Z}dz_1-Zdz_2)=\frac{1}{\sqrt{2B}}(\bar{Z}dz_1+Zd\bar{z}_1-Zd\bar{Z}), \qquad (6.98)\\
d\xi &= \frac{1}{\sqrt{2B}}(d\bar{Z}\wedge dz_1+dZ\wedge d\bar{z}_1-dZ\wedge d\bar{Z}).
\end{aligned}$$

Equation (6.90) implies that $d\xi\wedge\xi=dv\wedge\omega_2+O(v)$, and this relation is not satisfied by (6.98). Indeed, $d\xi\wedge\xi$ has a contribution

$$(Zd\bar{Z}-\bar{Z}dZ)dz_1d\bar{z}_1=dvdz_1d\bar{z}_1-2\bar{Z}dZdz_1d\bar{z}_1 \tag{6.99}$$

which is not proportional to dv. Then we conclude that the surface defined by (6.97) does not satisfy the relation (6.90), and thus it leads to a singular geometry.

Coming back to the relations (6.95), (6.96), we arrive at a conclusion any quadratic function (6.93) satisfying the regularity conditions (6.87) can be transformed by holomorphic reparameterizations into one of the following expressions:

$$v_I=z_1\bar{z}_1+z_2\bar{z}_2-B, \quad v_{II}=z_1\bar{z}_1-z_2\bar{z}_2-B \tag{6.100}$$

This demonstrates that the relations (6.87) are very restrictive.

Notice that exclusion of the wall (6.97) from the set of regular solutions is very important for correspondence between regular droplets and probe branes. Indeed, suppose (6.97)

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gave an allowed shape of the droplet. Then, sending B to zero, one could collapse the droplet to a curve $z_1 = -\bar{z}_2$ which must carry some amount of D3-brane charge. However, from the analysis of section 6.4, we know that such object does not exist (recall that probe branes must follow holomorphic curves). This is the key difference between the present situation and the case of 1/2-BPS bubbles discussed in [20]: there the branes looked like pointlike sources and there were no restrictions on the shape of the droplets.

We will now demonstrate that exclusion of surface (6.97) is a part of a general pattern: by collapsing a regular droplet in 1/4-BPS case, one always arrives at a holomorphic curve. This fact provides a nontrivial correspondence between the regular supergravity solutions and brane probe analysis presented in section 6.4.

Collapsing droplets and holomorphic curves. Let us consider a family of regular surfaces $v_\epsilon(z_a, \bar{z}_a)$ parameterized by ϵ, and assume that at $\epsilon = 0$ equation $v_\epsilon(z_a, \bar{z}_a) = 0$ describes a two-dimensional curve rather than a three-dimensional surface. Assuming that $v_\epsilon(z_a, \bar{z}_a)$ is a smooth function of ϵ and that relations (6.87) are satisfied for any $\epsilon > 0$, we will demonstrate that the two dimensional curve $v_0(z_a, \bar{z}_a) = 0$ must be holomorphic.

Near any point on the $v_0(z_a, \bar{z}_a) = 0$ curve, one can always make a holomorphic change of coordinates and rewrite the equation $v_0 = 0$ as

$$z_2 = f(z_1, \bar{z}_1).$$

Since surfaces $v_\epsilon(z_a, \bar{z}_a) = 0$ must surround this curve, at small ϵ one can always write

$$v_\epsilon(z_a, \bar{z}_a) = |z_2 - f(z_1, \bar{z}_1)|^2 - |\epsilon h(z_a, \bar{z}_a)|^2 + O(\epsilon^4). \tag{6.101}$$

Moreover, in a vicinity of the surface $v_\epsilon = 0$, coordinates $(z_2, \bar{z}_2)$ can be eliminated from function h.

Let us first assume that $\det(\partial_a \bar{\partial}_b v_0) \neq 0$. Then one can take $\epsilon \to 0$ limit in conditions (6.87). While doing this, it is important to introduce a scaling $\lambda \sim \epsilon^{-2}$, otherwise the matrix in the lhs of (6.87) would have a non-vanishing determinant. Introducing a finite $\tilde{\lambda} = \epsilon^2 \lambda$, we find an equation:

$$M_{a\bar{b}} \equiv \partial_a \bar{\partial}_b v_0 + (\epsilon^{-2} \tilde{\lambda} \partial_a v \bar{\partial}_b v|_{v=0})_{\epsilon=0} = g \partial_a w \bar{\partial}_b \bar{w} + O(v_0) \tag{6.102}$$

For small values of ϵ, the surface $v_\epsilon = 0$ can be parameterized by $z_1, \bar{z}_1$ and a pure phase η:

$$z_2 = f(z_1, \bar{z}_1) + \epsilon \eta \; h(z_1, \bar{z}_1), \qquad \bar{\eta}\eta = 1, \tag{6.103}$$

so we can compute the derivatives:

$$\epsilon^{-1} \bar{\partial}_1 v_\epsilon|_{v=0} = -\bar{\partial}_1 f \bar{\eta} \bar{h} - \bar{\partial}_1 \bar{f} \eta h + O(\epsilon), \qquad \epsilon^{-1} \bar{\partial}_2 v_\epsilon|_{v=0} = \eta h + O(\epsilon). \tag{6.104}$$

Substituting these expressions into (6.102) and taking the derivative of the left-hand side, we find an expression for $\tilde{\lambda}(z_1, \bar{z}_1, \eta)$. Factorizing (6.102) for this value of $\tilde{\lambda}$, we find

$$g^{1/2} dw = A_a(z_1, \bar{z}_1, \eta) dz^a + O(v_0). \tag{6.105}$$

Since $z_2 = f(z_1, \bar{z}_1)$ on the $v_0 = 0$ curve, the last relation is only possible if eta-dependence factorizes in A_a: $A_a(z_1, \bar{z}_1, \eta) = F(z_1, \bar{z}_1, \eta)\tilde{A}_a(z_1, \bar{z}_1)$. This means that, up to an overall coefficient, matrix $M_{a\bar{b}}$ is eta-independent.

Substituting the expressions (6.104) into the definition (6.102) of $M_{a\bar{b}}$ and keeping track of the η-dependence and factors of $\tilde{\lambda}$, one can schematically write $M_{a\bar{b}}$ as

$$M_{a\bar{b}} = \begin{pmatrix} \partial_1\bar{\partial}_1 v_0 + \tilde{\lambda}(a_0 + a_1\eta^2)(\bar{a}_0 + \bar{a}_1\bar{\eta}^2) & \partial_1\bar{\partial}_2 v_0 + \tilde{\lambda}\bar{b}(a_0 + a_1\eta^2) \\ \partial_2\bar{\partial}_1 v_0 + \tilde{\lambda}b(\bar{a}_0 + \bar{a}_1\bar{\eta}^2) & \partial_2\bar{\partial}_2 v_0 + \tilde{\lambda}b\bar{b} \end{pmatrix}. \tag{6.106}$$

Function $\tilde{\lambda}$ is determined by solving the equation $\det(M_{a\bar{b}}) = 0$. Substituting the resulting value of $\tilde{\lambda}$ back into (6.106), and requiring $M_{1\bar{1}}/M_{1\bar{2}}$ to be eta-independent, one concludes that $a_1 = 0$. This implies that $\bar{\partial}_1 f = 0$, i.e. f is a holomorphic function. Such conclusion falsifies our original assumption that $\det(\partial_a\bar{\partial}_b v_0) \neq 0$, so the matrix $\partial_a\bar{\partial}_b v_0$ has to be degenerate.

The condition $\det(\partial_a\bar{\partial}_b v_0) = 0$ can be rewritten as an equation for $f(z_1, \bar{z}_1)$:

$$-\partial_1\bar{\partial}_1 f(\bar{z}_2 - \bar{f}) - \partial_1\bar{\partial}_1\bar{f}(z_2 - f) + \partial_1\bar{f}\bar{\partial}_1 f = 0 \tag{6.107}$$

Restricting this relation to the curve $v_0 = 0$, we find that f must be holomorphic. This means that the droplet collapses to a curve

$$z_2 = f(z_1), \tag{6.108}$$

Thus, by utilizing small–ϵ analysis, we have shown that the droplets described by equation (6.101) can only be regular if function $f(z_1, \bar{z}_1)$ is holomorphic, this implies that regular droplets can only collapse to holomorphic curves. This conclusion is in a perfect agreement with discussion of sections 6.3 and 6.4, where both probe analysis and consistency of SUGRA were used to demonstrate that supersymmetric D3-branes must follow holomorphic profiles.

Summary. Let us summarize the results of this long subsection. By requiring the geometries (6.2) to be regular, we arrived at the following picture for the boundary conditions in the $y = 0$ hyperplane. This Kahler space is divided into a set of droplets, where Kahler potential satisfies one of the two conditions:

$$y = 0: \quad \begin{array}{l} Z = -\frac{1}{2}, \ \partial_a\bar{\partial}_b K(z, \bar{z}, y = 0) = 0, \\ \\ Z = +\frac{1}{2} \end{array} \tag{6.109}$$

Notice that the relation $\partial_a\bar{\partial}_b K(z, \bar{z}, y = 0) = 0$ is crucial in enforcing regularity inside the $Z = -\frac{1}{2}$ droplets, and, as we will demonstrate in the next subsection, it is also needed to uniquely specify the solution of the Monge-Ampere equation.

Moreover, in contrast to the boundaries of 1/2-BPS droplets, which can be arbitrary [20], the domain walls in 1/4-BPS case must obey a restriction coming from regularity. In particular, we demonstrated that there exists a real function $v(z_a, \bar{z}_a)$

which defines the boundaries between droplets (via an equation $v = 0$) and satisfies the differential relations (6.87):

$$\partial_a \bar{\partial}_b v + \lambda \partial_a v \bar{\partial}_b v = g \partial_a w \bar{\partial}_b \bar{w} + O(v), \quad \det(\partial_a v \partial_b w)|_{v=0} \neq 0. \tag{6.110}$$

Here $w(z_1, z_2)$ is a holomorphic function. Looking at few examples, we showed that (6.110) gives a nontrivial restriction on the shapes of the droplets. In particular, it was demonstrated that regular droplets can only collapse to holomorphic curves, in a perfect agreement of the D brane analysis presented in sections 6.3), (6.4.

6.6 Asymptotic behavior and perturbative expansion

In the previous subsection we discussed the behavior of the solutions near $y = 0$ hyperplane and found that regularity imposes nontrivial boundary conditions on the Kahler potential. However, behavior at $y = 0$ cannot fix the solutions on Monge-Ampere equation uniquely: one also has to specify Kahler potential at infinity. In this subsection we will discuss examples of large R behavior which lead to metrics with interesting asymptotics, and we will demonstrate that, once the large-distance behavior is fixed, any combination of the boundary conditions (6.64), (6.68) leads to the unique solution of Monge-Ampere equation.

The metrics which are most interesting from the point of view of AdS/CFT approach $AdS_5 \times S^5$ at large values of y. As discussed in section 6.2, $AdS_5 \times S^5$ can be embedded into the general 1/4-BPS ansatz in two different ways, so, to describe an asymptotically-AdS space, a Kahler potential should approach either (6.5) or (6.14) at large values of y. It turns out that in both cases a more natural way to impose asymptotic boundary conditions is to formulate them at large values of $R = \sqrt{r^2 + y^2}$[32] rather than y:

$$K_{(6.5)} = -\frac{1}{2} y^2 \log y + \frac{r^2}{2} + O(\log R), \tag{6.111}$$

$$K_{(6.14)} = \frac{1}{2} y^2 \log y + \frac{y^2}{2} \log(1 - x_2^2) + \frac{1}{2} \log R + o(\log R). \tag{6.112}$$

We also recall that $r \to \infty$ has different geometric interpretations for (6.111) and (6.112): in the first case one goes to infinity of flat four-dimensional space, while in the second case large values of r correspond to the boundaries and to the ends of the cylinder (see figure 2b). It is useful to introduce special notation for the leading terms in (6.111) and (6.112):

$$K_I \equiv -\frac{1}{2} y^2 \log y + \frac{r^2}{2}, \quad K_{\rm II} \equiv \frac{1}{2} y^2 \log y + \frac{y^2}{2} \log(1 - x_2^2) + \frac{1}{2} \log R. \tag{6.113}$$

These Kahler potentials correspond to $\mathbf{C}^2$ with $Z = \frac{1}{2}$ and to a strip with $Z = -\frac{1}{2}$. Let us now perturb these solutions.

If some finite region with $Z = -\frac{1}{2}$ is added to the geometry described by K_I, the asymptotic behavior would remain unchanged, so at large distances one can treat the effects of the insertion as perturbation. In other words, we can write

$$K = K_I + K^{(1)}, \tag{6.114}$$

[32]We recall that r was introduced as a radial coordinate in the two-dimensional Kahler space, and, generically being ambiguous, such coordinate is well-defined for spaces whose Kahler potential asymptotes to (6.5) or (6.14).

and at large distances (6.2) reduces to a linear equation for $K^{(1)}$:

$$\Delta_{z,\bar{z}}K^{(1)} = 2y^{-1}\partial_y K^{(1)} - y\partial_y(y^{-1}\partial_y K^{(1)}). \tag{6.115}$$

While this equation should not be trusted near $y = 0$ (where K_I and $K^{(1)}$ become comparable), one can formally extend the perturbation theory in the entire $y \geq 0$ region, keeping in mind that the series would converge only for large values of y. Such extension will allow us to count the number of degrees of freedom and to show that the boundary conditions (6.64), (6.68) specify the solution uniquely (at least in perturbation theory).

We begin with selecting some finite region D of $y = 0$ hypersurface and requiring that

$$y\partial_y(y^{-1}\partial_y K^{(1)}) = 2 \tag{6.116}$$

there (this corresponds to setting $Z = -\frac{1}{2}$ in for (6.114)). Rewriting equation (6.115) in terms of $H = y\partial_y(y^{-1}\partial_y K^{(1)})$:

$$\Delta_{z,\bar{z}}H + y^{-1}\partial_y(y\partial_y H) + \frac{4}{y^2}H = 0, \quad H|_{y=0} = \begin{cases} 2, & (z,\bar{z}) \in D \\ 0, & (z,\bar{z}) \notin D, \end{cases} \tag{6.117}$$

one can find a unique solution which vanished at infinity. However, there is still an ambiguity in function $K^{(1)}$: since at infinity we only require $K^{(1)} \ll K_I \sim R^2$, any harmonic function $\tilde{K}^{(1)}(z,\bar{z}) = o(R^2)$ would lead to a solution with correct asymptotics:

$$K^{(1)} = \int_\infty^y ydy \int_\infty^y \frac{dy}{y}H + \tilde{K}^{(1)}(z,\bar{z}): \qquad \Delta_{z,\bar{z}}\tilde{K}^{(1)} = 0, \quad \frac{K^{(1)}}{K_I} \stackrel{R\to\infty}{\longrightarrow} 0. \tag{6.118}$$

Notice that this freedom in choosing $K^{(1)}$ is crucial for ensuring that the second regularity condition in (6.64) can be imposed in region D: to cancel a non-zero contribution of function K_I at $y = 0$, $(z,\bar{z}) \in D$, one needs some ambiguity in $\tilde{K}^{(1)}$. Moreover, the harmonic function $\tilde{K}^{(1)}$ and the unwanted contribution K_0 (which must satisfy the homogeneous Monge-Ampere equation (6.63)) to the Kahler potential have the same amount of freedom,[33] so it appears that the ambiguities in (6.118) and in its higher-order counterparts can be used to remove K_0, and, once this is done, the perturbative expansion of function K would be fixed uniquely.

To fix the ambiguity in (6.118), we will require $K^{(1)}$ to be analytic in the region D and to vanish on all boundaries ∂D. Due to the maximum principle, this uniquely determines harmonic function $\tilde{K}^{(1)}(z,\bar{z})$ in the compact D, and we will set $\tilde{K}^{(1)} = 0$ on the complement of this region. Once function $K^{(1)}$ is determined, one can repeat the analysis for higher orders in perturbation theory, and again the ambiguity can be fixed order by order:

$$K^{(p)}|_{\partial D, y=0} = 0, \quad p > 1. \tag{6.119}$$

[33]To demonstrate this, one can consider a *formal* perturbation theory in (6.63) by writing $K_0 = \frac{1}{2}z_a\bar{z}_a + \epsilon\hat{K}(z,\bar{z})$, truncating (6.63) to the first order in ϵ, and formally setting $\epsilon = 1$. Then one finds that $\hat{K}$ has the same number of free parameters as a harmonic function. Of course, this argument is very heuristic, and one should not take the *solution* of the truncated equation seriously, however the *number of degrees of freedom* will not be affected by the higher-order terms.

While we only expect the perturbation series to converge at large values of y, the Kahler potential can be analytically continued to the entire $y > 0$ subspace, and the result would satisfy the Monge-Ampere equation (6.2) as well as boundary conditions which were imposed order by order:

$$-\frac{y}{2}\partial_y(\frac{\partial_y K}{y})\Big|_{y=0} = \begin{cases} -\frac{1}{2}, \ (z,\bar{z}) \in D \\ \ \ \frac{1}{2}, \ (z,\bar{z}) \notin D \end{cases}, \quad K|_{\partial D, y=0} = 0, \quad K = K_I + O(\log R). \tag{6.120}$$

Assuming that function $K_0 \equiv \lim_{y\to 0} K$ remains regular inside the region D, we arrive at the equation (6.63) along with a boundary condition:

$$(z,\bar{z}) \in D: \quad \partial_1\bar{\partial}_1 K_0 \partial_2\bar{\partial}_2 K_0 - \partial_1\bar{\partial}_2 K_0 \partial_2\bar{\partial}_1 K_0 = 0, \qquad K_0|_{\partial D, y=0} = 0 \tag{6.121}$$

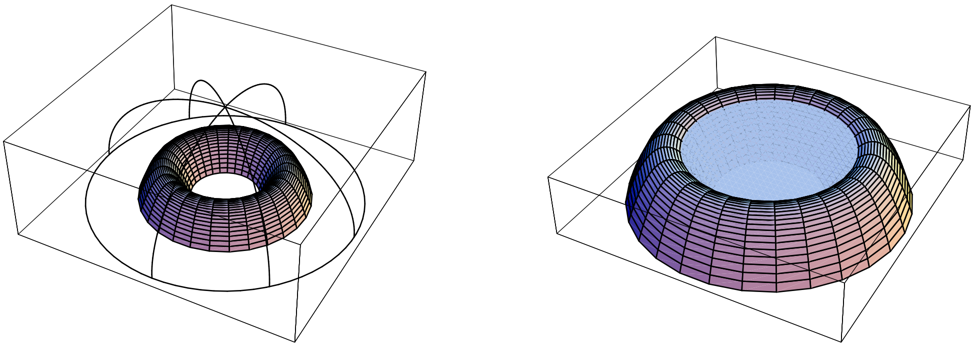
We will now demonstrate that function K_0 vanishes in the region D, so the boundary condition (6.64) is satisfied.

As we discussed before, the homogeneous Monge-Ampere equation can be easily solved in terms of some holomorphic coordinate w: $K_0 = K_0(w,\bar{w})$, then it is convenient to perform a holomorphic reparameterization: $(z_1, z_2) \to (w, v)$. Assuming that this change of variables is regular inside D, we conclude that the image of D in $(w, v, \bar{w}, \bar{v})$ space is compact. Starting with an arbitrary point $(w_0, v_0) \in D$, one can consider a complex plane $w = w_0$. Due to compactness of D, this plane must intersect the boundary ∂D along some hypersurface, then $K_0(w_0, \bar{w}_0) = 0$. Since $(w_0, v_0) \in D$ was arbitrary, we conclude that $K_0|_{\partial D} = 0$ implies $K_0|_D = 0$. This statement can be interpreted as a "maximum principle" for the homogeneous Monge-Ampere equation. Of course, our arguments were rather heuristic, but they can be made precise.

To summarize, we showed that starting with solution K_I whose boundary conditions correspond to $\mathbf{C}^2$ with $Z = \frac{1}{2}$, and introducing an arbitrary distribution of compact droplets with $Z = -\frac{1}{2}$, one can use perturbation theory to construct a solution which satisfies boundary condition (6.64) inside the droplets and condition (6.68) outside. This implies that, for any distribution of droplets, perturbation theory leads to a unique regular geometry. Notice that the differential restriction in (6.64) was crucial both for enforcing regularity and for ensuring uniqueness of the solution. A similar perturbation theory can be developed around solution (6.112), in this case one introduces compact droplets with $Z = \frac{1}{2}$ and requires $\partial_a\bar{\partial}_b K_0$ to vanish in the exterior of the droplets. It would also be interesting to study geometries with more exotic asymptotics: in the 1/2-BPS case, where the system was exactly solvable [20], such solutions were discussed in [33].

6.7 Topology and charges

The bubbling solutions discussed in this section have regular metrics and source-free field strengths, so, to allow non-zero fluxes, the geometries must have nontrivial topology. In this subsection we will explore the topological structure of 1/4-BPS solutions and show that they indeed contain some non-contractible five-cycles. We will also demonstrate that the integrals of F_5 over such cycles give non-zero answers, and discuss some global restrictions on the distributions of the droplets imposed by quantization of charge.



(a) (b)

Figure 7: Topology of the 1/4 geometries. To construct a non-contractible cycle, one should start with a four-dimensional "cap" which ends on a three-dimensional surface surrounding a compact droplet (a), and fiber ψ over the cap. The projection of the cap onto the Kahler space fills the interior of the sphere depicted in figure (a). Alternatively, one can start with a cap ending on a curve inside $Z = -\frac{1}{2}$ region, and fiber the three-sphere over it. The projection of the cap onto the Kahler space looks like a "membrane" depicted in figure (b).

To analyze the topology of the solution (6.2), we recall that either S^3 or S^1 collapses at $y = 0$ hyperplane. This leads to two simple constructions of non-contractible five-cycles.

I. If there exists a compact region with $Z = -\frac{1}{2}$ (i.e. with collapsing S^3) in $y = 0$ hyperplane, then such droplet can be surrounded by a three-dimensional surface $\mathcal{S}$ which lies entirely in the $Z = \frac{1}{2}$ region (see figure 7a). Constructing a four-dimensional "cap", which goes to $y > 0$ and has $\mathcal{S}$ as its boundary, and fibering ψ over it, one arrives at a five-dimensional surface Ω. Notice that, since radius of ψ-direction goes to zero on $\mathcal{S}$, the surface Ω is compact: restricting the metric (6.2) to Ω in a vicinity of $\mathcal{C}$, one finds a regular geometry without a boundary:[34]

$$ds_5^2 = ds_{10}^2|_\Omega = g^{-1}\left[ds_3^2 + dy^2 + y^2 d\psi^2\right]. \tag{6.122}$$

Moreover, this construction does not allow the surface $\mathcal{S}$ to move into $Z = -\frac{1}{2}$ region (ψ-direction does not shrink there), so the five-cycle Ω is non-contractible. An analogous "bubbling" effect was described in [20].

II. A construction of five-cycles containing S^3 is less-straightforward, since there are no compact droplets with $Z = -\frac{1}{2}$.[35]

To get some intuition, we consider the 1/2-BPS geometries of [20]. As discussed in section 6.2, these solutions can be embedded in the general 1/4-BPS ansatz, and the boundary between droplets is invariant under phase rotations of z_2 (see (6.35)).

[34] We wrote the four-dimensional Kahler metric appearing in (6.2) as $2\partial_a\bar{\partial}_b K_0 dz^a d\bar{z}^b = ds_3^2 + dx_\perp^2$, where $x_\perp$ is a direction orthogonal to $\mathcal{S}$.

[35] One might think that the spherical droplet (6.21) arising from $AdS_5 \times S^5$ has a compact interior, but, since it touches the boundary of the space ($|z_2| = 1$), this droplet cannot be surrounded by any three-dimensional surface.

This allows a very simple pictorial representation of droplets in terms of coordinates $(z_1, \bar{z}_1, |z_2|^2)$, and one can argue that, while there is only one connected region with $Z = -\frac{1}{2}$, generically this region is not simply-connected (an example of map from (z_1, z_2) plane to the boundary of the droplet in $(z_1, \bar{z}_1, |z_2|^2)$ space is presented in figure 4). As was shown in [20], one can get a non-trivial five-cycle by taking a non-contractible curve $\mathcal{C}$ in $\tilde{Z} = \frac{1}{2}$ region, constructing a two-manifold which ends on $\mathcal{C}$, and fibering S^3 over it. To recover Ω in the context of 1/4-BPS geometries, we select a non-contractible curve $\mathcal{C}$ in $y = 0, z_2 = 0$ plane, construct a two dimensional surface ω which ends on $\mathcal{C}$ and explores $y > 0$, $z_2 = 0$ region, and fiber S^3 over ω. A surface ω and a curve $\mathcal{C}$ can be moved to non-zero values of z_2, but, as long as $\mathcal{C}$ stays inside the $Z = \frac{1}{2}$ droplet, the resulting five-cycle Ω has no boundary. Since the original curve $\mathcal{C}$ was non-contractible, the projection of Ω onto Kahler manifold covers a "hole" in the $Z = \frac{1}{2}$ droplet, so Ω is a non-contractible cycle. An example of two-dimensional surface ω is presented in figure 7b.

Using the 1/2-BPS example as a guide, we can propose a general way of getting five-cycles involving the S^3. If the region $Z = \frac{1}{2}$ is not simply-connected, it contains a non-contractible curve $\mathcal{C}$. Then one can build a two-dimensional surface ω which stays at positive values of y and has $\mathcal{C}$ as its boundary. Fibering the three-sphere over ω, one gets a non-contractible five-cycle Ω. Looking at the metric (6.2), one can see that Ω has no boundary at $y = 0$, so this five-cycle has a topology of S^5. It appears that the five-spheres I and II are the only topologically-nontrivial cycles in the geometries (6.2).

Since the geometries contain non-contractible five spheres, and a non-zero five-form is present in the solution, it is natural to evaluate the integrals F_5 over the cycles. The amount of flux is invariant under small deformations of cycles, so it is convenient to choose these surfaces to make the computation easier. The four-dimensional "cap" surrounding $Z = -\frac{1}{2}$ region can be deformed into a surface located at small values of y, then Ω is parameterized by $(z_a, \bar{z}_b)$ and ψ. The metric at small values of y has already been analyzed in section 6.5, so we find

$$\begin{aligned} ds_{10}^2 &= h^{-2}\left[-(dt+\omega)^2 + d\psi^2 + 2\partial_a\bar{\partial}_b K_1 dz^a d\bar{z}^b\right] + h^2(dy^2 + y^2 d\Omega_3^2) \\ F_5 &= -d[y^4 h^4 (dt+\omega)] \wedge d\Omega_3 + h^{-4} d\psi \wedge \det(\partial_a\bar{\partial}_b K_1) d^2z d^2\bar{z} + \dots \end{aligned} \tag{6.123}$$

The field strength contains additional terms at the same order, but they will be irrelevant for out computation. Integrating F_5 over the $Z = -\frac{1}{2}$ region, one finds a very simple expression for the flux in terms of the volume of the droplet:

$$\begin{aligned} ds_{10}^2 &= h^{-2}[-(dt+\omega)^2 + d\psi^2] + g_{ij}^{(4)} dx^i dx^j + h^2(dy^2 + y^2 d\Omega_3^2) \\ \int_{\Omega_5} F_5 &= 2\pi V_4 \equiv 2\pi \int \sqrt{g^{(4)}} d^4x. \end{aligned} \tag{6.124}$$

A counterpart of this formula for 1/2-BPS geometries was encountered in [20], where the agreement with field theory picture [29] was also demonstrated. Unfortunately the matrix model in the 1/4-BPS case is more complicated, but it would be nice to interpret (6.124) in terms of eigenvalues of holomorphic matrices X and Y.

Let us now turn to the type II cycles. Again, they can be moved to small values of y, but now one also has freedom in deforming the contour $\mathcal{C}$ inside $Z = -\frac{1}{2}$ bubble. Integrating the expression for F_5, we find the flux:

$$\int_{\Omega_5} F_5 = 4i\pi^2 \int_\omega \partial_a \bar{\partial}_b K dz^a d\bar{z}^b \Big|_{y=0} : \qquad \partial\omega = \mathcal{C}. \tag{6.125}$$

Notice that the contribution to the integral comes only from the part of ω which lies inside $Z = \frac{1}{2}$ region: $\partial_a \bar{\partial}_b K = 0$ otherwise. Moreover, it is clear that the integral in (6.125) is invariant under small deformations of ω.

To summarize, we found that the 1/4-BPS bubbling solutions have very interesting topological structure: they can contain two types of non-contractible five cycles, and the fluxes through such spheres have very simple geometrical meaning. Since the charges computed in (6.123), (6.125) must be quantized, one finds a set of global restrictions on the "volumes" of the droplets and two-cycles ω. It would be very nice to connect this data with field theory picture developed in [31].

6.8 Relation to brane webs

By looking at the explicit form of the Killing spinor for the geometry (6.2), one can see that the isometry ∂_t is rotational. Since the timelike Killing vector in flat space generates translational isometry, the system (6.2) cannot describe asymptotically-flat solutions. However, the flat asymptotics can arise as a result of certain singular limits and in this subsection we will discuss two interesting possibilities. The first one leads to standard D3 branes with flat worldvolume, while the second limit reproduces the geometry (5.8) produced by the webs of D3 branes.

The Killing spinor has a very simple time dependence ($\epsilon \sim e^{it/2}$), which can be removed by rescaling the t-coordinate: $t = \lambda^{-1}\tilde{t}$, and sending λ to infinity while keeping $\tilde{t}$ fixed. To keep $g_{\tilde{t}\tilde{t}}$ finite, we also need to rescale $h^{-2} = \lambda^2 \tilde{h}^{-2}$ and there are two natural ways to do it.

1. *The limit of flat branes.*

   Let us consider a shift $e^G = \lambda e^{\tilde{G}}$ and send λ to infinity, while keeping $\tilde{G}$ finite. To retain the canonical periodicity of ψ, we will also rescale $y = \lambda\tilde{y}$. Then the leading terms in $\frac{1}{\lambda}$ expansion become

$$h^{-2} = \lambda^2 \tilde{y} e^{\tilde{G}}, \quad Z = \frac{1}{2} - \lambda^{-2} e^{-2\tilde{G}}, \quad K = \lambda^2 K_{-1}(z) + K_0(y) + \lambda^{-2} K_1. \tag{6.126}$$

   Equation relating Kahler potential and Z implies that

$$y^{-1}\partial_y K_0 = -\log\tilde{y} + C \tag{6.127}$$

To make the metric finite, we rescale the coordinates on S^3, then, dropping tildes, we arrive at the solution:

$$\begin{aligned} ds_{10}^2 &= ye^G dx_{1,3}^2 + (ye^G)^{-1}\left[2\partial_a\bar{\partial}_b K_{-1} dz^a d\bar{z}^b + dy^2 + y^2 d\psi^2\right], \\ \det h_{a\bar{b}} &= \frac{1}{4}. \end{aligned} \tag{6.128}$$

This geometry describes a two-dimensional Calabi-Yau manifold and a set of branes whose distribution is governed by function $H = (ye^G)^{-2}$. In this order the equation for the metric decouples and we have to look at the equation of motion for F_5 to conclude that H is harmonic. The branes are occupying points in the transverse space.

2. *D3 branes wrapping holomorphic cycles.*

   An alternative flat limit can be obtained by a rescaling which keeps the radius of the sphere finite: $e^G = \lambda^{-1} e^{\tilde{G}}$, $y = \lambda \tilde{y}$. Then we find

$$\begin{aligned} h^{-2} &= \lambda^2 y e^{-G}, \quad Z = -\frac{1}{2} + \lambda^{-2} e^{2G}, \quad \psi = \lambda^{-1}\tilde{\psi}, \quad t = \lambda^{-1}\tilde{t} \\ ds^2 &= ye^{-G}(-dt^2 + d\psi^2) + y^{-1}e^G\left[dy^2 + y^2 d\Omega_3^2\right] + 2y^{-1}e^{-G}\partial_a\bar{\partial}_b K dz^a d\bar{z}^b \end{aligned} \tag{6.129}$$

   The Kahler potential is not rescaled and it satisfies the following equations:

$$\begin{aligned} K &= \lambda^2 K_{-1}(y) + K_0, \quad \partial_y K_{-1} = y\log y, \quad e^{2G} = -\frac{1}{2} y\partial_y(y^{-1}\partial_y K_0), \\ \det\, h_{a\bar{b}} &= \lambda^{-1} y^2 e^{2G} W\bar{W} \quad \to \quad \det\, h_{a\bar{b}} = \frac{1}{4} y^2 e^{2G} \end{aligned} \tag{6.130}$$

   The gauge $W = \frac{1}{2}\lambda^{1/2}$ was chosen to avoid a singularity in the metric. To remove an explicit y-dependence of the metric, we define $H = y^{-1}e^G$ and rescale the Kahler potential: $K_0 = y^2\tilde{K}$. Then we arrive to the following geometry:

$$\begin{aligned} ds^2 &= H^{-1}(-dt^2 + d\psi^2) + H\left[dy^2 + y^2 d\Omega_3^2\right] + 2H^{-1}\partial_a\bar{\partial}_b\tilde{K} dz^a d\bar{z}^b \\ H^2 &= -\frac{1}{2} y^{-3}\partial_y(y^3\partial_y\tilde{K}), \qquad \det\, h_{a\bar{b}} = \frac{1}{4} H^2 \end{aligned} \tag{6.131}$$

   This solution goes over to (5.8), (4.3), (4.4) if one identifies H with $e^{-3A/2}$. The expressions for the RR five-form also agree.

6.9 Summary

Let us summarize the results of this section. While the local description of 1/4-BPS bubbling geometries (6.2) has been discussed before [19], the allowed boundary conditions were not known. In particular, a recent proposal by [23] appeared to be incomplete since it clearly disagreed with expectations from the probe analysis (by shrinking the droplets of [23], one could arrive at sources which are not allowed in string theory). By requiring the geometries of [19] to be regular at the hyperplane $y = 0$, we found an improved version of the picture proposed in [23]: the hyperplane is divided into regions where one of the conditions (6.109) should be satisfied. Moreover, in contrast to 1/2-BPS case, the shapes of the droplets cannot be arbitrary, but rather they are described by an equation $v(z_a, \bar{z}_a) = 0$, and function v must satisfy the relations (6.110). We demonstrated that these restrictions on v are consistent with results of probe analysis which requires the brane profiles to be holomorphic. We also showed that the allowed locations of D3 branes can be determined either from open string computations (which reduces to a DBI analysis) or from consistency of supergravity equations (this amounts to a description in terms of close strings), and there is a perfect agreement between these independent results. As a by-product of the probe analysis, we gave a clear geometric interpretation of Mikhailov's description of giant gravitons [24].

Once the distribution of droplets in $y = 0$ plane is specified, one can try to solve a nonlinear Monge-Ampere equation to construct the corresponding geometry. While we were not able to find new explicit solutions, we used perturbation theory to demonstrate that, for a fixed asymptotic behavior, conditions (6.109) specify the solution uniquely, and any allowed distribution of droplets leads to a regular geometry without sources. All fluxes are geometric: we showed that solutions contain non-contractible five-cycles and evaluated fluxes through them (see equations (6.124), (6.125)).

We also considered some explicit examples of 1/4-BPS geometries, in particular we showed that one can embed $AdS_5 \times S^5$ into an ansatz (6.2) in two different ways, and we also embedded all 1/2-BPS solutions constructed in [20]. Finally, by taking a limit of bubbling solution (6.2), we recovered the web of D3 branes which was discussed in section 5.

7. 1/4-BPS geometries in M theory

In the previous section we discussed 1/4-BPS geometries with $AdS_5 \times S^5$ asymptotics. The interest in such metrics is driven by their relation to supersymmetric states in four dimensional field theory. However, there are other important cases of AdS/CFT correspondence and understanding of eleven dimensional bulk configurations might shed some light on the dynamics of $(2, 0)$ six-dimensional CFT and of conformal theory on M2 brane. In this section we will discuss 1/4-BPS geometries in M theory. As we will see, they share many properties of their ten-dimensional counterparts, in particular, the discussion of droplets and their boundaries would essentially mimic the arguments presented in the previous section. It is interesting to notice that, while in the 1/2-BPS case the equation governing ten-dimensional geometries was much easier than its M theory counterpart [20], the Monge-Ampere equations describing the 1/4-BPS geometries appear to have the same degree of difficulty in ten and eleven dimensions.

We will look at a particular set of geometries which preserve SO(6) symmetry. They could correspond either to supersymmetric states in (2,0) theory on $R \times S^5$, or to some space-dependent configurations in three-dimensional CFT. It might be useful to recall that 1/2-BPS solutions of [20] had $\mathrm{SO}(6) \times \mathrm{SO}(3)$ symmetry, so now we are breaking half of the supercharges which were used to produce "translations" along SO(3).

7.1 Local structure of the solution

As already mentioned, we want to study eleven dimensional geometries which preserve 8 supercharges as well as bosonic SO(6) symmetry. Fortunately, the local structure of the solution can be easily extracted from the results of [18].

We recall that authors of [18] constructed the most general solution of eleven dimensional supergravity which contains AdS_5 factor:[36]

$$\begin{aligned} ds^2 &= 4e^{2\lambda}ds^2_{\rm AdS} + e^{-4\lambda}\left(h_{ij}dx^i dx^j + \frac{dy^2}{\cos^2\zeta}\right) + \frac{4e^{2\lambda}}{9}\cos^2\zeta(d\psi+\rho)^2 \\ F_4 &= -(\partial_y e^{-6\lambda})V_4 + \frac{1}{\cos^2\zeta}(*_4 d_4 e^{-6\lambda})dy - \frac{4}{9}\cos^4\zeta(*_4\partial_y\rho)(d\psi+\rho) \\ &\quad + \left[\frac{4}{9}\cos^2\zeta *_4 d_4\rho - \frac{4}{3}e^{-6\lambda}J\right]dy(d\psi+\rho). \end{aligned} \tag{7.1}$$

As shown in [18], the four dimensional metric h_{ij} is Kahler and there are various differential relations between the metric components and the Kahler form J:

$$\begin{aligned} y &= e^{3\lambda}\sin\zeta, \quad \rho = J\cdot d_4\left[\frac{1}{2}\log(\cos^2\zeta\sqrt{h})\right], \quad \partial_y J = -\frac{2}{3}y d_4\rho \\ \partial_y\log\sqrt{h} &= -3y^{-1}\tan^2\zeta - 2\partial_y\log\cos\zeta \end{aligned} \tag{7.2}$$

Notice that, as a consequence of supersymmetry, ∂_ψ turns out to be a Killing vector for the geometry (7.1). It is convenient to introduce complex coordinates z_a, $\bar z_a$ then the relations between metric, Kahler form and Kahler potential become especially simple:

$$h_{ij}dx^i dx^j = 2h_{a\bar b}dz^a d\bar z^b, \quad J = ih_{a\bar b}dz^a\wedge d\bar z^b, \quad h_{a\bar b} = \partial_a\bar\partial_b K \tag{7.3}$$

The geometries with SO(6) symmetries can be obtained from (7.1) by performing the following analytic continuation:

$$ds^2_{\rm AdS}\to -d\Omega_5^2, \quad e^\lambda\to ie^\lambda, \quad \zeta\to i\zeta, \quad \psi\to t, \tag{7.4}$$

and the resulting solution reads:

$$\begin{aligned} ds^2 &= -\frac{4e^{2\lambda}}{9}\cosh^2\zeta(dt+\rho)^2 + 4e^{2\lambda}d\Omega_5^2 + e^{-4\lambda}\left(2h_{a\bar b}dz^a d\bar z^b + \frac{dy^2}{\cosh^2\zeta}\right), \\ F_4 &= (\partial_y e^{-6\lambda})V_4 - \frac{1}{\cosh^2\zeta}(*_4 d_4 e^{-6\lambda})dy - \frac{4}{9}\cosh^4\zeta(*_4\partial_y\rho)(dt+\rho) \\ &\quad + \left[\frac{4}{9}\cosh^2\zeta *_4 d_4\rho + \frac{4}{3}e^{-6\lambda}J\right]dy(dt+\rho). \end{aligned} \tag{7.5}$$

Various functions appearing in this solution satisfy a system of differential equations:[37]

$$\begin{aligned} &\sinh\zeta = ye^{-3\lambda}, \quad \rho = \frac{3i}{2}(\partial D - \bar\partial D), \qquad y^{-1}\partial_y J = 2i\partial\bar\partial D, \\ &\partial_y D = y^{-1}\tanh^2\zeta = \frac{ye^{-6\lambda}}{1+y^2e^{-6\lambda}}, \qquad e^{3D} = 4\cosh^2\zeta\ \det h_{a\bar b}. \end{aligned} \tag{7.6}$$

[36] To avoid unnecessary complications and to make contact with notation introduced in [20], we set $m=\frac{1}{2}$ in the formulas of [18] and rewrite the solution in terms of AdS space with unit radius.

[37] We introduced a convenient function D whose normalization is chosen to agree with results of [20] for the 1/2-BPS case.

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Considering the equation for the y-dependence of the Kahler potential K, we find a relation between K and D:

$$\partial\bar{\partial}\left[y^{-1}\partial_y K - 2D\right] = 0. \tag{7.7}$$

Since metric is not affected by addition of an (anti)holomorphic function to Kahler potential, we can choose a gauge where

$$D = \frac{1}{2y}\partial_y K. \tag{7.8}$$

At this point the entire solution is completely specified by the Kahler potential (although some expressions are simpler in terms of D):

$$e^{-6\lambda} = \frac{y^{-1}\partial_y D}{1 - y\partial_y D}, \quad \tanh^2\zeta = y\partial_y D, \quad \rho = \frac{3i}{2}(\partial - \bar{\partial})D, \quad D = \frac{1}{2y}\partial_y K. \tag{7.9}$$

The last remaining equation relates D and $\det h_{a\bar{b}}$:

$$4\det h_{a\bar{b}} = (1 - y\partial_y D)e^{3D} = \left[1 - \frac{1}{2}y\partial_y(y^{-1}\partial_y K)\right]\exp\left(\frac{3}{2y}\partial_y K\right). \tag{7.10}$$

This is an M theory counterpart of the Monge-Ampere equation (6.2).

To summarize, the most general eleven-dimensional solution preserving 8 supercharges and SO(6) isometry is given by (7.5), it is completely specified by the Kahler potential K (see (7.9)) which satisfies a Monge-Ampere equation (7.10). In the remaining part of this section we will study the system (7.5), (7.9) (7.10) in more detail. We begin with discussing some known solutions which are covered by the general ansatz (7.5).

7.2 Examples

The simplest supersymmetric geometries with SO(6) symmetry are $AdS_4 \times S^7$ and $AdS_7 \times S^4$, so it seems natural to discuss them first. However, to avoid repetition, we will begin with embedding a more general class of solutions preserving only 16 supercharges, and then come back to $AdS_p \times S^q$.

7.2.1 1/2-BPS bubbling solutions

The most general 1/2-BPS solution of M theory was constructed in [20], where it was shown that the metric and fluxes are uniquely determined in terms of one function $\tilde{D}(z, \bar{z}, x)$ which satisfies a continual Toda equation:

$$\partial_x^2 e^{\tilde{D}} + 4\partial_z\partial_{\bar{z}}\tilde{D} = 0. \tag{7.11}$$

The explicit form of the solution is reviewed in the appendix B.2 (see equation (B.21)), here we will only need to recall that the 1/2-BPS metrics have SO(6) $\times$ SO(3) isometries and coordinate x is related to the radii of five– and three-spheres in a very simple way: $x = \frac{1}{2}R_2 R_5^2$. Moreover, regularity restricts the allowed boundary conditions for the normal derivative of $\tilde{D}$ at $x = 0$:

$$x = 0: \quad \begin{array}{l} \partial_x\tilde{D} = 0, \ \tilde{D} - \text{finite} \\ \partial_x\tilde{D} = \frac{1}{x} + O(1) \end{array} \tag{7.12}$$

Thus the entire $x = 0$ plane is divided into two types of regions with arbitrary curves separating them (see figure 3).

In the appendix B.2 we embed the 1/2-BPS solutions into the more general geometry (7.5), (7.9), (7.10) which is parameterized by one function $D(z_a; \bar{z}_a; y)$. The relation between 1/2-BPS and 1/4-BPS variables is given by (B.34):

$$D(z, We^{-i\phi}; \bar{z}, We^{i\phi}; y) = \tilde{D}(z, \bar{z}; x), \quad y = x\cos\theta, \quad W = x\sin\theta e^{-\tilde{D}}. \tag{7.13}$$

Here coordinates θ and ϕ parameterize the two-sphere in the 1/2-BPS solution:

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta \left[d\left(\phi - \frac{2t}{3}\right)\right]^2. \tag{7.14}$$

To find interpretation of the boundary conditions (7.12), we recall the relation (B.38):

$$\partial_y D = \frac{\cos\theta \partial_x \tilde{D}}{1 - \sin^2\theta x \partial_x \tilde{D}}, \qquad x = 0: \quad \begin{array}{l} \partial_x \tilde{D} = 0 \to \partial_y D = 0 \\ \partial_x \tilde{D} = \frac{1}{x} \to \partial_y D = \frac{1}{y} \end{array} \tag{7.15}$$

Now it is natural to impose boundary conditions at the hypersurface where $y = x\cos\theta = 0$: it consists of the regions described by the last equation as well as points where $\cos\theta = 0$ and $\partial_y D = 0$. Thus we see that the lift of 1/2-BPS geometries gives the solutions of (7.5), (7.9), (7.10), and for regularity the $y = 0$ surface is divided into regions with two types of boundary conditions:

$$y = 0: \qquad \begin{array}{c} \mathrm{I}:\ \partial_y D = \frac{1}{y} + O(1) \\ \mathrm{II}:\ \partial_y D = 0,\ D - \text{finite} \end{array} \tag{7.16}$$

As we will see in section 7.3, regularity also imposes some restrictions on the derivatives of Kahler potential in the region I.

The boundary between the regions can be found by repeating the logic which led to (6.35):

$$z_2 \bar{z}_2 = e^{-2\hat{D}(z,\bar{z})}, \quad \hat{D} = \tilde{D} - \log x. \tag{7.17}$$

While there are many similarities in the description of 1/2-BPS solutions in IIB string theory and in the eleven-dimensional supergravity, on the technical level Toda equation (7.11) is much more complicated than its IIB counterpart (6.23). In particular, for the Laplace equation one can easily write down an explicit solution corresponding to the most general boundary condition (6.24), while it is not clear how to do so for the system (7.11), (7.12). While we believe that the appropriate solution does exist for any distribution of droplets, only few explicit solutions are know, and now we will discuss their embedding into the 1/4-BPS ansatz in more detail.

7.2.2 pp-wave and $AdS_p \times S^q$

The simplest solution of the Toda equation corresponds to the eleven-dimensional pp-wave [20]:

$$x = \frac{1}{4}u^2 v, \quad z - \bar{z} = i\left(\frac{u^2}{2} - v^2\right), \quad e^{\tilde{D}} = \frac{u^2}{4} \tag{7.18}$$

Using (7.13), we can introduce the variables appropriate for the 1/4-BPS case:

$$y = \frac{1}{4}u^2 v \cos\theta, \quad z_2 = v \sin\theta e^{-i\phi}, \quad z_1 - \bar{z}_1 = i\left(\frac{u^2}{2} - v^2\right), \quad e^D = \frac{u^2}{4}. \tag{7.19}$$

It is easy to see that the boundary conditions (7.16) are satisfied on the surface $y = 0$, and one can extract the equations for the surface separating regions I and II. To do so, we observe that region I corresponds to $u = 0$, where

$$i(z_1 - \bar{z}_1) = v^2 \geq v^2 \sin^2\theta = z_2 \bar{w}$$

Thus the boundary between two regions is a three-dimensional surface

$$\text{Im } z_1 = -\frac{1}{2} z_2 \bar{z}_2. \tag{7.20}$$

Once function D is known, one can easily recover the Kahler potential integrating the last equation in (7.9). In the case of the pp-wave the simplest way to proceed is to eliminate v, θ from the expression for y:

$$y = \frac{u^2}{4}\sqrt{\frac{u^2}{2} - Z - W^2}, \quad W = |z_2|, \quad Z = -i(z_1 - \bar{z}_1)$$

This leads to the Kahler potential

$$\begin{aligned} K &= \int^{u^2/4} d\tilde{u} \log\tilde{u}[6\tilde{u}^2 - 2\tilde{u}(Z + W^2)] \\ &= \frac{\tilde{u}^2}{6}(-4\tilde{u} + 3(Z + W^2) + 6(2\tilde{u} - Z - W^2)\log\tilde{u})\Big|_{\tilde{u}=\frac{u^2}{4}}. \end{aligned}$$

This expression is not very illuminating, we presented its derivation just to illustrate the general procedure.

Next we look at $AdS_7 \times S^4$. In this case the solution of Toda equation is parameterized by the radial coordinate r of AdS and one of the coordinates α of the sphere [20]:

$$e^D = \frac{r^2 L^{-6}}{4 + r^2}, \quad z = \left(1 + \frac{r^2}{4}\right)\cos\alpha e^{i\psi}, \quad 4x = L^{-3} r^2 \sin\alpha. \tag{7.21}$$

The coordinates for 1/4-BPS embedding are

$$y = \frac{r^2}{4L^3}\sin\alpha\cos\theta, \quad z_1 = \left(1 + \frac{r^2}{4}\right)\cos\alpha e^{i\psi}, \quad z_2 = L^3\left(1 + \frac{r^2}{4}\right)\sin\alpha\sin\theta e^{-i\phi}, \tag{7.22}$$

and the region I corresponds to $r = 0$ where

$$z_1 \bar{z}_1 = \cos^2\alpha \leq 1 - L^{-6} z_2 \bar{z}_2.$$

Thus the boundary between regions corresponds to an ellipsoid

$$z_1 \bar{z}_1 + L^{-6} z_2 \bar{z}_2 = 1. \tag{7.23}$$

This equation reduces to (7.20) if one writes $z_1 = i + L^{-6}\tilde{z}_1$ and takes L to infinity.

The $\mathrm{AdS}_4\times\mathrm{S}^7$ solution works in a similar way:

$$e^D = 4L^{-6}\sqrt{1+\frac{r^2}{4}}\sin^2\alpha, \qquad z_1 = e^{i\psi}\left(1+\frac{r^2}{4}\right)^{1/4}\cos\alpha, \qquad 2x = L^{-3}r\sin^2\alpha,$$
$$y = \frac{1}{2}L^{-3}r\sin^2\alpha\cos\theta, \qquad z_2 = \frac{L^3}{4}\frac{r}{\sqrt{r^2+4}}\sin\theta e^{-i\phi}, \tag{7.24}$$

the region I now corresponds to $\alpha = 0$:

$$16L^{-6}z_2\bar{z}_2 \leq \frac{r^2}{r^2+4} = 1 - \frac{1}{(z_1\bar{z}_1)^2}.$$

and the boundary between regions is given by the surface

$$16L^{-6}z_2\bar{z}_2 = 1 - \frac{1}{(z_1\bar{z}_1)^2} \tag{7.25}$$

This reduces to (7.23) if we make a holomorphic reparameterization $z_1 \to z_1^{-1/2}$ and a constant rescaling of W. Notice that this conformal map exchanges the interior and exterior of the circle $|z_1| = 1$. This observation is consistent with the fact that $\mathrm{AdS}_7\times\mathrm{S}^4$ and $\mathrm{AdS}_4\times\mathrm{S}^7$ had complementary boundary conditions in $x = 0$ plane.

To summarize, we have shown that 1/2-BPS geometries of [20] fit nicely into more general solution (7.5), (7.9) (7.10), and regular geometries must satisfy the boundary conditions (7.16)[38] with some additional restrictions on the surfaces separating I and II regions. We also presented explicit examples of such surfaces which came from the $\mathrm{AdS}_p\times\mathrm{S}^q$ and plane wave geometries. In the next subsection we will demonstrate that conditions (7.16) are required for regularity even for the most general 1/4-BPS solution (which cannot be embedded into the ansatz of [20]) and we will also derive the restrictions on the shapes of the droplets.

7.3 Boundary conditions

As in the IIB case, local equations (7.5), (7.9) (7.10) have to be supplemented by some boundary conditions. The five-sphere contracts along the hypersurface where $e^\lambda = 0$ and, since we want to keep g_{tt} finite, this implies that $y = 0$. However, the $y = 0$ hypersurface has another region where $\zeta = 0$ and e^λ does not vanish. Let us consider regularity conditions which should be imposed on these two subsets.

I. If $e^\lambda \to 0$, then to keep nonzero g_{tt}, one should send ζ to infinity. In the vicinity of such points it is convenient to parameterize the leading contribution to the metric in

[38] As we will see in the next subsection, regularity also imposes an extra restriction on the values of Kahler potential in $y = 0$ hyperplane. It can be viewed as a requirement on the "integration constant' in equation (7.8).

terms of a new function $f = ye^{-2\lambda}$ which remains finite:

$$\begin{aligned} ds^2 &= -\frac{4f^2}{9}(dt+\rho)^2 + \frac{4}{f}\left[yd\Omega_5^2 + \frac{dy^2}{4y}\right] + 2\frac{f^2}{y^2}h_{a\bar{b}}dz^a d\bar{z}^b, \\ e^D &= ye^{\tilde{D}}, \quad \rho = \frac{3i}{2}(\partial - \bar{\partial})\tilde{D}, \quad K = \int 2y\log y\, dy + \int dy^2\tilde{D} + K_0(z,\bar{z}), \\ e^{-6\lambda} &= -\frac{1}{y^3\partial_y\tilde{D}}, \qquad\qquad f = |\partial_y\tilde{D}|^{-1/3}. \end{aligned} \tag{7.26}$$

This metric describes regular geometry if function $\tilde{D}$ remains finite as y goes to zero and

$$\partial_a\bar{\partial}_b K_0 = 0, \qquad \det(\partial_a\bar{\partial}_b\tilde{D})|_{y=0} \neq 0, \quad \partial_y\tilde{D} < 0. \tag{7.27}$$

It is easy to see that the last two requirements are satisfied, so in this case the boundary condition reduces to

$$y = 0: \quad \partial_y D = \frac{1}{y} + O(1), \quad \partial_a\bar{\partial}_b K = 0. \tag{7.28}$$

II. Assuming that $e^{2\lambda}$ remains finite, we find the following expansions:

$$\begin{aligned} \sinh\zeta &= ye^{-3\lambda}, \quad \partial_y D = ye^{-6\lambda}, \quad K = K_0 + y^2K_1 + y^4K_2, \\ ds^2 &= -\frac{4e^{2\lambda}}{9}(dt+\rho)^2 + 4e^{2\lambda}d\Omega_5^2 + e^{-4\lambda}\left(2h_{a\bar{b}}dz^a d\bar{z}^b + \frac{dy^2}{\cosh^2\zeta}\right). \end{aligned} \tag{7.29}$$

This metric is regular as long as K_0 is a non-singular Kahler potential. Looking at the right-hand side of equation (7.10), we observe that this is indeed the case as long as D remains finite. Thus we arrive at the boundary condition:

$$y = 0: \quad \partial_y D = 0, \quad D - \text{finite}. \tag{7.30}$$

III. The regularity conditions are slightly more involved near the hypersurfaces which separate regions I and II. A similar problem for the IIB geometries was analyzed in section 6.5, and now the intuition developed there will be applied to the eleven-dimensional case.

Shapes of the droplets. As we saw, the hyperplane $y = 0$ is divided into regions where either $e^{2\lambda}$ or $y^2e^{-4\lambda}$ go to zero, so both factors should vanish near the wall separating different domains. This suggests natural coordinates in a vicinity of a point on the wall:

$$X = e^{2\lambda}, \quad Y = ye^{-2\lambda}. \tag{7.31}$$

In particular, this change of variables leads to the following relations:

$$y = X^2Y, \quad e^{2\lambda}\cosh^2\zeta = X^2 + Y^2 \equiv R^2. \tag{7.32}$$

As one approaches a wall, function R goes to zero, and we require that the metric (7.5):

$$ds^2 = -\frac{4e^{2\lambda}}{9}\cosh^2\zeta(dt+\rho)^2 + 4e^{2\lambda}d\Omega_5^2 + e^{-4\lambda}\left(h_{ij}dx^idx^j + \frac{dy^2}{\cosh^2\zeta}\right) \tag{7.33}$$

remains regular at the points where $R = 0$. Starting with relation between y and X, Y, one can always parameterize the Kahler metric by $v(X, Y)$ and three more coordinates $\tilde{x}_a$ which are orthogonal to it. As we will see, regularity conditions impose certain restrictions on function $v(z_a, \bar{z}_b)$.

Near a point where $R = 0$, the metric (7.33) becomes singular unless the five-dimensional sphere combines with radial coordinate X to give a flat six-dimensional space ($ds_6^2 = 4dX^2 + 4X^2d\Omega_5^2$), and the metric in remaining five directions describes a flat space $R^{1,4}$. Subtracting ds_6^2 from (7.33) and keeping only the leading terms in powers of R, we find:

$$\begin{aligned} ds_5^2 &= -\frac{4}{9}(X^2+Y^2)(dt+\rho)^2 + \left[-4dX^2 + \frac{dy^2}{X^2(X^2+Y^2)}\right] + X^{-4}h_{ij}dx^idx^j \\ &= -\frac{4}{9}(X^2+Y^2)(dt+\rho)^2 + \left[dY^2 - \frac{(2XdX - YdY)^2}{(X^2+Y^2)}\right] + X^{-4}h_{ij}dx^idx^j \end{aligned} \tag{7.34}$$

This five-dimensional space should be orthogonal to ds_6^2, so, after the Kahler metric is rewritten in terms of $(v[X, Y], \tilde{x}_a)$, the contributions proportional to dX and to dX^2 should disappears from the last expression. This can only happen if v is a function of one variable $2X^2 - Y^2$, and, without loss of generality, we can set

$$v = 2X^2 - Y^2 = 2e^{2\lambda} - y^2e^{-4\lambda}. \tag{7.35}$$

Treating this relation as a cubic equation for $e^{2\lambda}$, one concludes that this warp factor depends on the Kahler base only through v (i.e. $e^{2\lambda} = f(v, y)$), then, integrating equations (7.6), (7.8) for D and K, one finds that Kahler potential is also a function of v and y only.[39] Of course, just as in the IIB case, this reduction of happens only in the vicinity of the wall. Starting with $K(v, y)$, we deduce Kahler metric and one-form ρ:

$$\begin{aligned} h_{ij}dx^idx^j &= 2\partial_v^2K|\partial v|^2 + 2\partial_vK\partial\bar{\partial}v = 2\partial_vK\partial\bar{\partial}v + \frac{\partial_v^2K}{2}\left(dv^2 + |(\partial-\bar{\partial})v|^2\right) \\ \rho &= \frac{3i}{4y}\partial_y\partial_vK(\partial-\bar{\partial})v \equiv \frac{3}{2y}\partial_y\partial_vK\eta, \qquad \eta = \frac{i}{2}(\partial-\bar{\partial})v. \end{aligned} \tag{7.36}$$

Substituting these expressions into (7.34) and requiring the resulting space to be regular, we conclude that[40]

$$\begin{aligned} \partial_v^2K &= \frac{X^4}{2R^2} + O(R^3), \quad \partial_y\partial_vK = \frac{X^2Y}{R^2} + O(R^2), \quad R^2 \equiv X^2 + Y^2, \\ ds_5^2 &= -\frac{2}{3}\eta dt + dY^2 + \frac{2}{X^4}\partial_vK\partial\bar{\partial}v + \lambda_1dv^2 + \lambda_2\eta^2. \end{aligned} \tag{7.37}$$

[39]There is also an "integration constant" $K_0(z_a, \bar{z}_a)$, and, just as in IIB case, but we will not discuss this further.

[40]To arrive at (7.37), one should follow the steps which led to (6.84), (6.86).

In the leading order in R, coefficients λ_1 and λ_2 must be (y, v)-independent. Moreover, extracting the contribution to $\partial_v K$ from equations for $\partial_v^2 K$ and $\partial_y \partial_v K$, we conclude that the leading contribution to $X^{-4}\partial_v K = \frac{1}{2} + O(R)$ is a constant. To describe regular geometry, the metric (7.37) should correspond to a flat five-dimensional space:

$$ds_5^2 = -\frac{2}{3}\eta dt + dY^2 + \partial\bar{\partial}v + \lambda_1 dv^2 + \lambda_2 \eta^2 = dY^2 + dS^2_{R^{1,3}}, \tag{7.38}$$

then, repeating the steps which led to (6.87), we find a restriction on function v:

$$\partial_a \partial_b v + \lambda \partial_a v \bar{\partial}_b v = g \partial_a w \bar{\partial}_b \bar{w} + O(v), \quad \det(\partial_a v \partial_b w)|_{v=0} \neq 0 \tag{7.39}$$

We conclude that a decomposition of $y = 0$ hyperplane into regions I and II produces a regular geometry if and only if the function v defining the boundaries of the droplets[41] obeys the relations listed in (7.39). The same differential conditions restrict the boundary of 1/4-BPS droplets in IIB supergravity (see equation (6.87)), and their consequences were discussed in great detail in section 6.5. Here we only mention that some necessary (but not sufficient) conditions can be formulated as *algebraic* relations (6.88), (6.89) between λ and derivatives of v:

$$\det(\partial_a \bar{\partial}_b v + \lambda \partial_a v \bar{\partial}_b v) = O(v), \qquad \det(\partial_a \bar{\partial}_b v)\big|_{v=0} \neq 0. \tag{7.40}$$

As in section 6.6, one can show that introducing the droplets whose shapes satisfy (7.39), imposing the boundary conditions (7.28), (7.30), and specifying asymptotic behavior of Kahler potential, one arrives at the unique solution of the Monge-Ampere equation, which leads to a regular geometry.

One can also discuss probe membranes and M5 branes on the 1/4-BPS geometries (7.5), and, following the arguments presented in sections 6.3, 6.4, it can be shown that the probe analysis and consistency of supergravity lead to the same requirements for the brane profiles. Moreover, the restrictions (7.40) prevent droplets from shrinking to branes which are excluded by the probe analysis (see similar discussion in section 6.5).

To analyze topology of the solutions (7.5), one needs to make minor modifications in the discussion of section 6.7: geometry (7.5) contains non-contractible four-cycles, which surround the regions with collapsing S^5, and seven-cycles, which are constructed by fibering S^5 over two-cycle in $\partial_y D = 0$ region (see figure 7). It is easy to see that such four– and sever-spheres carry nontrivial fluxes, but explicit expressions for $\int F_4$ and $\int F_7$ are not very illuminating.

8. Unified description of bubbling solutions

Looking back at discussions in the last two sections, one observes that, while there are striking similarities in the descriptions of 1/4-BPS states in ten and eleven dimensions,

[41]We recall that the domain walls separating the regions are defined by $v(z_a, \bar{z}_a) = 0$.

the boundary conditions look somewhat different: going from IIB to M theory, one interchanges Dirichlet and Neumann boundary conditions. A similar difference was also encountered in [20] for the 1/2-BPS solutions. In this section we will introduce an alternative parameterization of IIB solutions which makes the boundary conditions identical to those encountered in eleven dimensions. The analysis of this section is inspired by the matching 1/2-BPS solutions with 1/4-BPS ansatz: as discussed in the appendix B.1, the "improved" variables for IIB SUGRA arise naturally in the process of embedding. We will begin with discussion of 1/2-BPS solution.

1/2-BPS case. We begin with recalling the bubbling geometries in M theory [20]:

$$\begin{aligned} ds^2 &= -4e^{2\lambda}\cosh^2\xi(d\tau+V)^2 + \frac{e^{-4\lambda}}{\cosh^2\xi}\left[dx^2+e^D dzd\bar z\right] + 4e^{2\lambda}d\Omega_5^2 + x^2e^{-4\lambda}d\Omega_2^2 \\ F_4 &= d\left[-4x^3e^{-6\lambda}(d\tau+V) + 2\,\overset{*}{_3}\left\{e^{-D}x^2\partial_x\left(\frac{\partial_x e^D}{x}\right)dx + x\partial_x d_2 D\right\}\right]\wedge d^2\Omega \\ e^{-6\lambda} &= \frac{\partial_x D}{x(1-x\partial_x D)},\quad V=\frac{i}{2}(dz\partial_z - d\bar z\partial_{\bar z})D,\quad \sinh\xi = xe^{-3\lambda}. \end{aligned} \tag{8.1}$$

The solutions are parameterized by a function D satisfying Toda equation (7.11) and Neumann boundary conditions (7.12):

$$x=0:\quad \begin{cases} \partial_x D = 0,\ D - \text{finite}, \\ \partial_x D = \frac{1}{x} + O(1). \end{cases} \tag{8.2}$$

This should be contrasted with situation in IIB SUGRA, where 1/2-BPS solutions are parameterized by a harmonic function $\tilde Z(z,\bar z,x)$ (see (6.22), (6.23)), which satisfies the Dirichlet boundary conditions [20]:

$$\tilde Z(z,\bar z,x=0) = \pm\frac{1}{2}. \tag{8.3}$$

However, as we saw in section 6.2, one can also describe ten-dimensional solutions in terms of function D which has Neumann boundary conditions (8.2) (see equations (6.26), (6.30)):

$$\begin{aligned} ds^2 &= -e^{2\lambda}\cosh^2\xi(dt+V)^2 + \frac{1}{e^{2\lambda}\cosh^2\xi}(dx^2+dzd\bar z) + e^{2\lambda}d\Omega_3^2 + x^2e^{-2\lambda}d\tilde\Omega_3^2, \\ F_5 &= -\frac{1}{4}d\left[x^4e^{-4\lambda}(dt+V) - *_3\left\{x^2\partial_x\left(\frac{\partial_x D}{x}\right) + x\partial_x d_2 D\right\}\right]\wedge d\tilde\Omega_3 + dual, \\ e^{-4\lambda} &= \frac{\partial_x D}{x(1-x\partial_x D)},\quad V=-i(dz\partial_z - d\bar z\partial_{\bar z})D,\quad \sinh\xi = xe^{-2\lambda}. \end{aligned} \tag{8.4}$$

The systems (8.1) and (8.4) look strikingly similar, and the boundary conditions (8.2) are identical in both cases. The differences between (8.1) and (8.4) stem from the fact that in IIB case the time-like Killing vector is translational, while in M theory it is rotational, so function D obeys different equations (compare (6.27) and (7.11)). In the systems without fluxes, the relation between translational (rotational) Killing vector and Laplace (Toda) equation has been extensively discussed in the past [34, 35].

1/4-BPS case. Given the similarities between 1/2-BPS geometries in ten and eleven dimensions, it is interesting to see whether they persist for the 1/4-BPS solutions as well. To determine this, we rewrite the IIB solutions (6.2) in terms of a new function D, and compare the results with (7.5), (7.9), (7.10). Motivated by the discussion of section 6.2, we extend the relation (6.29) to arbitrary 1/4-BPS geometries by making a definition:

$$D = \frac{1}{2}(y^{-1}\partial_y K + \log y). \tag{8.5}$$

Then equation for Z (6.2) implies that

$$Z = -y\partial_y D + \frac{1}{2}. \tag{8.6}$$

Substituting this into (6.2) and introducing $e^{2\lambda} = ye^G$, one finds the following metric:

$$\begin{aligned} ds_{10}^2 &= -e^{2\lambda}\cosh^2\zeta(dt+\omega)^2 + e^{-2\lambda}\left[2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b + \frac{dy^2}{\cosh^2\zeta} + y^2 d\psi^2\right] + e^{2\lambda}d\Omega_3^2, \\ \omega &= i(\bar{\partial}-\partial)K, \quad e^{-4\lambda} = \frac{y^{-1}\partial_y D}{1-y\partial_y D}, \quad \sinh\zeta = ye^{-2\lambda}. \end{aligned} \tag{8.7}$$

The Monge-Ampere equation also simplifies in terms of D:

$$\det h_{a\bar{b}} = \frac{1}{4}(1-y\partial_y D)e^{2D}. \tag{8.8}$$

We observe that similarities between the type IIB system (8.5)–(8.8) and its M theory counterpart ((7.5), (7.9), (7.10)) are even more striking than in the 1/2-BPS case. Perhaps this is related to the fact that now Killing vectors are rotational in both cases.

9. Discussion

Since the results of this paper have already been summarized in the introduction, in this section we will discuss some applications and open problems.

While D branes can be described in terms of either open or closed strings, traditionally one utilizes open string picture to find the positions of the branes, and then uses this information as an input for the SUGRA analysis. Along with earlier work [17], this article provides an evidence for plausibility of an alternative approach, where brane profiles are found directly in the closed-string picture. Then an agreement with open string analysis serves as a nontrivial check of the open/closed string duality. So far, such agreement was demonstrated only for configurations preserving eight or more supercharges, and it would be nice to see whether it persists for branes with lower supersymmetry. It would also be interesting to understand the nature of the agreement: since DBI and SUGRA descriptions are valid in different corners of the parameter space, one may unravel some new non-renormalization theorems.

While discussing bubbling geometries, we encountered a very interesting restriction (1.2) on the shape of the droplets, and it would be nice to acquire a more geometrical understanding of this conditions. Moreover, since the geometries discussed in section 6 correspond to 1/4-BPS states in a $\mathcal{N}=4$ SYM, one should be able to find a field-theoretic counterpart of (1.2). We recall that in the 1/2-BPS case, boundary conditions in supergravity [20] had a direct interpretation in terms of a matrix model on the field theory side [29]. Using this correspondence as a guide, one expects to see the condition (1.2) in a matrix model introduced in [31], but, to do so, a better understanding of the matrix model is required. In fact, the quantum mechanical system introduced in [31] describes 1/8-BPS states as well, and the most general gravity solution with this amount of supersymmetry was discussed in [36]. It would be very nice to understand regularity condition in this case. It is also very interesting to find the geometries preserving less than eight supercharges, since they might viewed as microscopic states contributing to an entropy of the black hole constructed in [37].

There are also open problems in the 1/4-BPS case. While the geometries (6.2) give the bulk description of local states in field theory, the metrics corresponding to 1/4-BPS non-local states are still not known. In the 1/2-BPS case, the geometries corresponding to various defects in IIB string [38] and M [39] theories turned out to be more complicated than the solutions of [20], and this trend is expected to continue for configurations with lower supersymmetry. However, the boundary conditions encountered in [38, 39] are as transparent as ones found in [20], so it would be interesting to find their 1/4-BPS counterparts. An additional motivation for such investigation comes from the fact that the brane probe analysis (i.e. an analog of [24]) has been already performed in [40].

Finally, the Monge-Ampere equations encountered in this article present an interesting technical challenge. From the probe analysis, we know that the sources can be freely superposed, this indicates that Monge-Ampere equations might have some hidden linear structure. If this is indeed the case, it would be very interesting to find the right variables which make the equations linear and lead to explicit solutions.

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A. Derivation of metrics produced by brane webs

To construct a geometry produced by a connected string web, one needs to find a supersymmetric background which fits into the following ansatz:

$$\begin{aligned} ds^2_{\rm IIB} &= -f_1 dt^2 + g_{ij}dx^i dx^j + f_2 d\Omega_6^2 \\ B + iC^{(2)} &= dt\wedge V_i dx^i, \qquad \tau\equiv C^{(0)} + ie^{-\Phi} = \tau(x_i). \end{aligned} \tag{A.1}$$

Rather than solving the equations for Killing spinors on this geometry, we will choose an alternative path and construct the metrics describing a U-dual system. To be more precise, we begin with smearing the web in one of the transverse directions and performing a T duality in this direction. The lift of the resulting IIA solution to M theory gives a

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configuration with $\mathrm{SO}(6)\times\mathrm{U}(1)^2\times\mathrm{U}(1)_t$ symmetry:

$$\begin{aligned} ds^2 &= -e^{2A}dt^2+e^{2C}(dw_1+\chi dw_2)^2+e^{2D}dw_2^2+g_{MN}dX^MdX^N+e^{2B}d\Omega_5^2 \\ G_4 &= dt\wedge W_\alpha\wedge dw^\alpha \end{aligned} \tag{A.2}$$

To find supersymmetric geometries, we should solve the equations for the Killing spinor and it turns out that, while an assumption of translational invariance in w_1 and w_2 leads to certain simplifications in these equations, the first few steps towards solving them rely only on $\mathrm{SO}(6)\times\mathrm{U}(1)_t$ isometries. Thus we begin with searching for the the most general solution consistent with latter symmetry:

$$\begin{aligned} ds^2 &= -e^{2A}dt^2+g_{\mu\nu}dX^\mu dX^\nu+e^{2B}d\Omega_5^2 \\ G_4 &= dt\wedge F_3 \end{aligned} \tag{A.3}$$

and in subsection A.4 we will analyze the additional restrictions imposed by the ansatz (A.2). Our first goal is to solve the equations for the Killing spinor:

$$\nabla_m\eta+\frac{1}{288}\left[-\frac{1}{2}\gamma_m\not{G}+\frac{3}{2}\not{G}\gamma_m\right]\eta=0. \tag{A.4}$$

Looking at the components of this equation along the isometry directions, one can produce the projectors which do not contain derivatives of the Killing spinor. We begin with projectors which correspond to time and sphere directions:

$$\nabla_t\eta-\frac{1}{144}\gamma_t\not{G}\eta=0: \qquad \frac{1}{2}\not{\partial}A\eta-\frac{1}{144}\not{G}\eta=0 \tag{A.5}$$

$$\nabla_a\eta+\frac{1}{288}\gamma_a\not{G}\eta=0: \qquad -\frac{i}{2}e^{-B}\Gamma_S\eta+\frac{1}{2}\not{\partial}B\eta+\frac{1}{288}\not{G}\eta=0 \tag{A.6}$$

To arrive at the second relation we introduced the standard invariant fermions on the odd-dimensional sphere with unit radius [41]:

$$\tilde{\nabla}_a\eta=-\frac{i}{2}\tilde{\gamma}_a\Gamma_S\eta \tag{A.7}$$

The projectors (A.5) and (A.6) can be combined to produce a relation which does not contain fluxes:

$$-i\Gamma_S\eta+\not{\partial}\left(B+\frac{A}{2}\right)\eta=0. \tag{A.8}$$

We can use the diffeomorphisms in five-dimensional space spanned by X^M to choose $y=e^{B+A/2}$ to be one of the coordinates and to parameterize the orthogonal subspace by $x_1,\dots x_4$. Then the "geometric" projector (A.8) determines the y-component of the metric:

$$y=e^{B+A/2},\qquad g_{\mu\nu}dX^\mu dX^\nu\equiv e^{-A}dy^2+g_{ij}dx^idx^j. \tag{A.9}$$

In this frame the projector (A.8) reduces to a very simple relation:

$$i\Gamma_S\eta-\Gamma_y\eta=0. \tag{A.10}$$

So far we only discussed the time and sphere components of (A.4), let us not look at the remaining seven equations. Using the relation (A.5), we can rewrite those components as

$$\begin{aligned}
\nabla_\mu\eta-\frac{1}{8}\gamma_\mu\not{\partial}A\eta+\frac{3}{4}\frac{1}{144}\not{G}\gamma_\mu\eta&=0:\\
e^{A/8}\nabla_\mu(e^{-A/8}\eta)+\frac{1}{8}\partial^\nu A\gamma_{\nu\mu}\eta-\frac{e^{-A}}{48}\Gamma_0\not{F}\gamma_\mu\eta&=0.
\end{aligned}\tag{A.11}$$

Convenient rescalings

$$\eta=e^{A/8}\tilde{\eta},\quad e_\mu^{\mathbf{a}}=\tilde{e}_\mu^{\mathbf{a}}e^{A/4}\tag{A.12}$$

lead to simplifications in the equation (A.11):

$$\tilde{\nabla}_\mu\tilde{\eta}-\frac{e^{-3A/2}}{48}\Gamma_0\tilde{\not{F}}\tilde{\gamma}_\mu\tilde{\eta}=0.\tag{A.13}$$

Notice that this relation, as well as two remaining projectors (A.5), (A.6), does not mix $\eta_+=(1+i\Gamma_0)\eta$ and $\eta_-=(1-i\Gamma_0)\eta$, so, without loss of generality, we can impose a projection

$$\Gamma_0\eta=i\eta.\tag{A.14}$$

Let us summarize what we have learned so far. Assuming only $\mathrm{SO}(6)\times\mathrm{U}(1)_t$ symmetry and introducing convenient coordinates, we showed that the eleven-dimensional geometry must have the form

$$\begin{aligned}
ds^2&=-e^{2A}dt^2+e^{A/2}\left[e^{-3A/2}dy^2+h_{ij}dx^idx^j\right]+y^2e^{-A}d\Omega_5^2\\
G_4&=dt\wedge F
\end{aligned}\tag{A.15}$$

and the Killing spinor must satisfy the following relations

$$\not{\partial}A\eta-\frac{ie^{-3A/2}}{18}\not{F}\eta=0,\tag{A.16}$$

$$\nabla_\mu\eta-\frac{ie^{-3A/2}}{48}\not{F}\gamma_\mu\eta=0,\tag{A.17}$$

$$\Gamma_0\eta=i\eta,\qquad i\Gamma_S\eta-\Gamma_y\eta=0.\tag{A.18}$$

in the *reduced* five-dimensional space:[42]

$$g_{\mu\nu}dX^\mu dX^\nu\equiv e^{-3A/2}dy^2+h_{ij}dx^idx^j\tag{A.19}$$

Before we proceed with analysis of these equations, let us count the preserved supersymmetries. Eleven-dimensional spinor η has 32 real components and two independent projections (A.18) reduce the number of components to 8. The projector (A.16) breaks

[42]From now on we will only use the metric which appears in the square brackets in (A.15), so we drop tildas in the equation (A.13). We will also use Greek letters to denote five-dimensional indices and Latin letters for the four-dimensional ones.

one half of the remaining supersymmetries, so it appears that we are dealing with 1/8-BPS configuration. However a closer inspection of the system (A.16)–(A.18) demonstrates that the same bosonic background preserves spinors with both signs in $\Gamma_0\eta = \pm i\eta$ (while other signs in (A.16)–(A.18) being adjusted accordingly), so, even though we will only look for a four-component spinor satisfying (A.16)–(A.18), the resulting geometries will be 1/4-BPS.

The system (A.16)–(A.18) can be viewed as a set of relations for a spinor in seven dimensions spanned by matrices $(\gamma_\mu, \gamma_t, \Gamma_S)$. These seven objects are not independent since the product of gamma matrices in eleven dimensions is equal to one:

$$\Gamma_0\Gamma_{1234}\Gamma_y\Gamma_S = 1. \tag{A.20}$$

Multiplying this relation by η and using projections (A.18), we conclude that

$$\Gamma_{1234}\eta = \eta. \tag{A.21}$$

The seven dimensional spinor appearing in (A.16)–(A.18) came from the reduction on a five-sphere, so it can have at most 8 independent components. The projectors (A.18) truncate the number of components to 2, and after enforcing (A.16) one should end up with a spinor which is parameterized by one real number. In the remaining part of this section we will discuss the properties of this one-component spinor in more detail.

A.1 Equations for the spinor bilinears

To find the restrictions on the five-dimensional metric (A.19), we will study the equations for the spinor bilinears. Since equations (A.16), (A.17) and their hermitean conjugates will be used extensively, we rewrite them here for future reference:

$$\nabla_\mu\eta - \frac{ie^{-3A/2}}{48}\not{F}_3\gamma_\mu\eta = 0, \qquad \nabla_\mu\eta^\dagger - \frac{ie^{-3A/2}}{48}\eta^\dagger\gamma_\mu\not{F}_3 = 0 \tag{A.22}$$

$$\not{\partial}A\eta - \frac{ie^{-3A/2}}{18}\not{F}_3\eta = 0, \qquad \eta^\dagger\not{\partial}A - \frac{ie^{-3A/2}}{18}\eta^\dagger\not{F}_3 = 0 \tag{A.23}$$

We begin with solving the equation for the scalar bilinear $\eta^\dagger\eta$ (the choice of integration constant fixes the normalization of the spinor):

$$\nabla_\mu(\eta^\dagger\eta) - \frac{18}{48}2\partial_\mu A\ \eta^\dagger\eta = 0: \qquad \eta^\dagger\eta = e^{3A/4}. \tag{A.24}$$

Next we look at tensor bilinear:

$$J_{\mu\nu} = \eta^\dagger\gamma_{\mu\nu}\eta. \tag{A.25}$$

The second projector in (A.18) implies that this tensor cannot have legs in y direction:

$$J_{\mu\nu}e^\mu_{\mathbf{y}} = 0: \qquad J \equiv \frac{1}{2}J_{\mu\nu}dX^\mu\wedge dX^\nu = \frac{1}{2}J_{mn}dx^m\wedge dx^n. \tag{A.26}$$

Let us compute the derivative of the tensor and the exterior derivative of the two-form:

$$\nabla_\mu J_{\nu\lambda} = \frac{ie^{-3A/2}}{48}\eta^\dagger \left(\gamma_{\nu\lambda} \not{F}_3 \gamma_\mu + \gamma_m \not{F}_3 \gamma_{kl}\right)\eta \tag{A.27}$$

$$= -\frac{3}{8}\eta^\dagger(\gamma_{\nu\lambda}\gamma_\mu \not{\partial} A + \not{\partial} A \gamma_\mu \gamma_{\nu\lambda})\eta + \frac{ie^{-3A/2}}{8}\eta^\dagger(\gamma_{\nu\lambda}F_{\mu\rho\sigma}\gamma^{\rho\sigma} + 6F_{\mu\rho\sigma}\gamma^{\rho\sigma}\gamma_{\nu\lambda})\eta$$

$$\nabla_{[\mu}J_{\nu\lambda]} = -\frac{9}{4}\partial_{[\mu}AJ_{\nu\lambda]} + \frac{ie^{-3A/2}}{8}\eta^\dagger(\gamma_{[\nu\lambda}F_{\mu]\rho\sigma}\gamma^{\rho\sigma} + F_{\rho\sigma[\mu}\gamma^{\rho\sigma}\gamma_{\nu\lambda]})\eta$$

We begin with simplifying the y component of the last relation:

$$\frac{1}{3}\partial_y J_{kl} = -\frac{3}{4}\partial_y A J_{kl} + \frac{ie^{-3A/2}}{24}\eta^\dagger\{\gamma_{kl},\gamma^{pq}\}F_{ypq}\eta + \frac{ie^{-3A/2}}{12}4F_{lky}\eta^\dagger\eta$$

$$= -\frac{1}{4}\partial_y A J_{kl} + \frac{ie^{-3A/4}}{3}F_{lky} \tag{A.28}$$

To eliminate the term with anticommutator we used the relations which can be obtained by combining the second projection in (A.18) with (A.16):

$$\partial_y A\eta - \frac{ie^{-3A/2}}{6}F_{ypq}\gamma^{pq}\eta = 0, \quad \eta^\dagger\partial_y A - \frac{ie^{-3A/2}}{6}\eta^\dagger F_{ypq}\gamma^{pq} = 0 \tag{A.29}$$

Equation (A.28) can be rewritten as a simple expression for the flux:

$$F_{kly} = i\partial_y(e^{3A/4}J_{kl}). \tag{A.30}$$

Motivated by this relation, we compute the four-dimensional components of the 3-form $d(e^{3A/4}J)$:

$$\nabla_{[m}(e^{3A/4}J_{kl]}) = -\frac{1}{4}e^{3A/4}\eta^\dagger\{\not{\partial} A, \gamma_{mkl}\}\eta + \frac{ie^{-3A/4}}{8}\eta^\dagger(\gamma_{[kl}F_{m]pq}\gamma^{pq} + F_{pq[m}\gamma^{pq}\gamma_{kl]})\eta$$

$$= -\frac{ie^{-3A/4}}{12}\left(F_{pq[k}\eta^\dagger\gamma^{pq}\gamma_{lm]}\eta - F_{pq[l}\eta^\dagger\gamma_k\gamma^{pq}\gamma_{m]}\eta + F_{pq[m}\eta^\dagger\gamma_{kl]}\gamma^{pq}\eta\right)$$

$$+\frac{ie^{-3A/4}}{8}\eta^\dagger(\gamma_{[kl}F_{m]pq}\gamma^{pq} + F_{pq[m}\gamma^{pq}\gamma_{kl]})\eta$$

$$= \frac{ie^{-3A/4}}{24}\left(F_{pq[m}\eta^\dagger\gamma^{pq}\gamma_{kl]}\eta + 2F_{pq[m}\eta^\dagger\gamma_l\gamma^{pq}\gamma_{k]}\eta + F_{pq[m}\eta^\dagger\gamma_{kl]}\gamma^{pq}\eta\right)$$

$$= \frac{ie^{-3A/4}}{6}\left(F_{p[km}\eta^\dagger\gamma^p\gamma_{l]}\eta - F_{p[lm}\eta^\dagger\gamma_{k]}\gamma^p\eta\right)$$

$$= \frac{i}{3}F_{lkm}$$

Combining this with (A.30), we arrive at the final expression for the flux:

$$F = id(e^{3A/4}J) \tag{A.31}$$

Alternatively, we can extract the flux from looking at a bilinear built out of the projectors (A.23):

$$\eta^\dagger\{\gamma_{\mu\nu\lambda}, \not{\partial} A\}\eta - \frac{i}{18}e^{-3A/2}\eta^\dagger\{\gamma_{\mu\nu\lambda}, \not{F}\}\eta = 0$$

$$6\partial_{[\mu}AJ_{\nu\lambda]} - \frac{i}{18}e^{-3A/2}(12F_{\mu\nu\lambda}e^{3A/4} + 18F_{\rho\sigma[\mu}\eta^\dagger\gamma_{\nu\lambda]}{}^{\rho\sigma}\eta) = 0.$$

Different components of the last equation give:[43]

$$\partial_y A J_{kl} = \frac{i}{3} e^{-3A/4} \left(F_{ylk} + \frac{1}{2} F_{pqy} \epsilon_{kl}{}^{pq} \right), \tag{A.32}$$

$$6\partial_{[m} A J_{kl]} = \frac{2i}{3} e^{-3A/4} (F_{mlk} + F_{pq[m} \epsilon_{kl]}{}^{pq}). \tag{A.33}$$

To simplify the second equation, we observe that

$$\epsilon^{klms} F_{pq[m} \epsilon_{kl]}{}^{pq} = \frac{1}{3} \epsilon^{klms} \epsilon_{kl}{}^{pq} F_{pqm} = \frac{2}{3} (g^{mp} g^{sq} - g^{mq} g^{sp}) F_{pqm} = 0. \tag{A.34}$$

This leads to a very simple expression for the four dimensional components of the flux:

$$F_{mkl} = 9ie^{3A/4} \partial_{[m} A J_{kl]} = 3ie^{3A/4} (dA \wedge J)_{klm}, \tag{A.35}$$

and, combining it with (A.31), we arrive at the equation for J:

$$0 = \left[d(e^{3A/4} J) - 3e^{3A/4} dA \wedge J \right]_{klm} = e^{3A} \left[d(e^{-9A/4} J) \right]_{klm} \tag{A.36}$$

Let us go back to the equation (A.32). First of all, it implies that J_{kl} is an anti-self-dual tensor. Further, by substituting the expression (A.30) for the flux into the right-hand side of that equation, we arrive at a useful relation between the y derivatives of the tensor bilinear:

$$\partial_y (e^{-3A/2} \epsilon_{kl}{}^{pq} J_{pq}) + e^{-3A/2} \partial_y (J_{pq}) \epsilon_{kl}{}^{pq} = 0. \tag{A.37}$$

The two-form J is especially useful since it is related to an almost complex structure. Indeed, we can use the Fierz identities to show that

$$J_{mp} J^{pn} = e^{3A/2} \delta_m^n \tag{A.38}$$

This implies that

$$\tilde{J}_m{}^n = ie^{-3A/4} J_m{}^n \tag{A.39}$$

is an almost complex structure on the four-dimensional manifold parameterized by x_m. In the next subsection we will demonstrate that this almost complex structure is integrable and we will also discuss a holomorphic two-form.

A.2 Complex structure and holomorphic two-form

To demonstrate that $\tilde{J}_m^n$ is a complex structure, we need to show that the Nijehuis tensor

$$N_{mn}{}^p = \tilde{J}_m{}^q \tilde{J}_{[n}{}^p{}_{,q]} - \tilde{J}_n{}^q \tilde{J}_{[m}{}^p{}_{,q]} \tag{A.40}$$

vanishes. We begin with recalling the four-dimensional components of the equation (A.27):

$$\begin{aligned} \nabla_m J_{kl} &= -\frac{3}{8} \eta^\dagger (\{\gamma_{klm}, \partial\!\!\!/ A\} + 2g_{m[l} [\gamma_{k]}, \partial\!\!\!/ A]) \eta + \frac{i}{8} e^{-3A/4} (4F_{mlk} + 2\epsilon_{kl}{}^{pq} F_{mpq}) \\ &= -\frac{9}{4} J_{[kl} \partial_{m]} A - \frac{3}{2} g_{m[l} J_{k]p} \partial^p A + \frac{i}{4} e^{-3A/4} (2F_{mlk} + \epsilon_{kl}{}^{pq} F_{mpq}). \end{aligned} \tag{A.41}$$

[43] Notice that the projector (A.21) implies that $\eta^\dagger \gamma_{klmp} \eta = \epsilon_{klmp} \eta^\dagger \eta$.

Taking antisymmetric part, we find

$$\nabla_{[m}J_{k]l} = -\frac{9}{4}J_{[kl}\partial_{m]}A - \frac{3}{4}g_{l[m}J_{k]p}\partial^p A + \frac{i}{4}e^{-3A/4}(2F_{mlk} - \epsilon_{l[k}{}^{pq}F_{m]pq}).$$

Notce that

$$\epsilon_{l[k}{}^{pq}F_{m]pq} = -\frac{1}{3}g_{lm}\epsilon_k{}^{rpq}F_{rpq}, \tag{A.42}$$

so we need to find the four-dimensional dual of the three-form. To this end we construct a bilinear using projectors (A.23):

$$\begin{aligned} 0 &= \eta^\dagger[\gamma_k, \not{\partial}A]\eta + \frac{i}{9}e^{-3A/2}F^{pqr}\eta^\dagger\gamma_{pqrk}\eta = 2\partial^p A J_{kp} + \frac{i}{9}e^{-3A/4}F^{pqr}\epsilon_{pqrk} \\ i\epsilon_{l[k}{}^{pq}F_{m]pq} &= -\frac{i}{3}g_{l[m}\epsilon_{k]}{}^{rpq}F_{rpq} = -6g_{l[m}\partial^p AJ_{k]p}e^{3A/4} \end{aligned} \tag{A.43}$$

Substituting this expression and the one for F_{klm} into (A.42), we find

$$\begin{aligned} \nabla_{[m}J_{k]l} &= -\frac{9}{4}J_{[kl}\partial_{m]}A - \frac{3}{4}g_{l[m}J_{k]p}\partial^p A + \frac{9}{2}\partial_{[m}AJ_{kl]} + \frac{3}{2}g_{l[m}\partial^p AJ_{k]p} \\ &= \frac{3}{4}\left[3J_{[kl}\partial_{m]}A + g_{l[m}J_{k]p}\partial^p A\right]. \end{aligned} \tag{A.44}$$

Let us rewrite this in terms of an almost complex structure $\tilde{J} = ie^{-3A/4}J$:

$$\begin{aligned} \nabla_{[m}\tilde{J}_{k]l} &= \frac{3}{4}\left[3\tilde{J}_{[kl}\partial_{m]}A + g_{l[m}\tilde{J}_{k]p}\partial^p A\right] - \frac{3}{4}\partial_{[m}A\tilde{J}_{k]l} \\ &= \frac{3}{8}\left[\tilde{J}_{kl}\partial_m A - \tilde{J}_{ml}\partial_k A + 2\tilde{J}_{mk}\partial_l A + g_{lm}\tilde{J}_{kp}\partial^p A - g_{lk}\tilde{J}_{mp}\partial^p A\right] \end{aligned}$$

To evaluate the Nijehuis tensor, we need to compute

$$\begin{aligned} \tilde{J}_k{}^q\nabla_{[m}\tilde{J}_{q]l} &= \frac{3}{8}\left[-g_{kl}\partial_m A - \tilde{J}_{ml}\tilde{J}_k{}^q\partial_q A + 2g_{mk}\partial_l A - g_{lm}g_{kp}\partial^p A - \tilde{J}_{kl}\tilde{J}_{mp}\partial^p A\right] \\ &= \frac{3}{8}\left[-g_{kl}\partial_m A - \tilde{J}_{ml}\tilde{J}_k{}^q\partial_q A + 2g_{mk}\partial_l A - g_{lm}\partial_k A - \tilde{J}_{kl}\tilde{J}_m{}^p\partial_p A\right] \end{aligned}$$

The right-hand side of this expression is symmetric under interchange of k and m, so we conclude that Nijehuis tensor vanishes. This implies that $\tilde{J}$ is an integrable complex structure and we can choose complex coordinates:

$$ds^2 = 2g_{a\bar{b}}dz^a d\bar{z}^b, \quad J = \frac{1}{2}J_{mn}dx^m \wedge dx^n = e^{3A/4}g_{a\bar{b}}dz^a \wedge d\bar{z}^b. \tag{A.45}$$

Notice that not only the four-dimensional space is complex, but it is also related to a Kahler space by a very simple rescaling. To see this we recall the equation (A.36) for the two-form and rewrite it in terms of the metric $g_{a\bar{b}}$:

$$d\left[e^{-3A/2}g_{a\bar{b}}dz^a \wedge d\bar{z}^b\right] \tag{A.46}$$

This implies that $e^{-3A/2}g_{a\bar{b}}$ is a Kahler metric and it can be written in terms of the potential $K(z, \bar{z}, y)$. In particular, we find:

$$g_{a\bar{b}} = e^{3A/2}\partial_a\bar{\partial}_b K. \tag{A.47}$$

At this point the solution is completely specified in terms of a real function A and Kahler potential K and the rest of this subsection will be devoted to finding a relation between them. To extract such relation, we define a new two-form

$$\Omega = \frac{1}{2}\eta^\dagger \gamma_{mn}\Gamma_{\overline{12}}\eta \; dx^{mn} \tag{A.48}$$

and compute its derivatives:

$$\nabla_\mu(\eta^\dagger\gamma_{\nu\lambda}\Gamma_{\overline{12}}\eta) - \frac{i}{48}e^{-3A/2}\eta^\dagger\left(\gamma_{\nu\lambda}\Gamma_{\overline{12}}\not{F}\gamma_\mu + \gamma_\mu\not{F}\gamma_{\nu\lambda}\Gamma_{\overline{12}}\right)\eta = 0. \tag{A.49}$$

To proceed it is convenient to introduce a holomorphic veilbein and flat gamma matrices:

$$g_{a\bar{b}} = \frac{1}{2}e_a^A e_{\bar{b}}^{\bar{B}}\delta_{A\bar{B}}, \qquad \{\Gamma_A, \Gamma_{\bar{B}}\} = \delta_{A\bar{B}}, \quad \{\Gamma_A, \Gamma_B\} = 0. \tag{A.50}$$

In particular, looking at various components of (A.45), we observe that

$$\eta^\dagger\Gamma_{AB}\eta = 0, \qquad \eta^\dagger[\Gamma_A, \Gamma_{\bar{B}}]\eta = \delta_{A\bar{B}}. \tag{A.51}$$

As we mentioned before, η should be viewed as a eight-component spinor in seven dimensional space and gamma matrices acting on this spinor are constrained by the relation (A.20). To proceed it is convenient to choose an explicit set of seven gamma matrices:

$$\begin{array}{lll} \Gamma_0 = i\sigma_3\otimes\sigma_3\otimes\sigma_3, & \Gamma_y = \sigma_1\otimes\sigma_3\otimes\sigma_3, & \Gamma_S = \sigma_2\otimes\sigma_3\otimes\sigma_3, \\ \begin{matrix}\Gamma_1\\ \Gamma_{\bar{1}}\end{matrix} = 1\otimes 1\otimes \begin{matrix}\sigma_-\\ \sigma_+\end{matrix}, & \begin{matrix}\Gamma_2\\ \Gamma_{\bar{2}}\end{matrix} = 1\otimes \begin{matrix}\sigma_-\\ \sigma_+\end{matrix}\otimes\sigma_3, & \end{array} \tag{A.52}$$

with the following actions of sigmas:

$$\sigma_3|\uparrow\rangle = |\uparrow\rangle, \quad \sigma_3|\downarrow\rangle = -|\downarrow\rangle, \quad \sigma_+|\downarrow\rangle = |\uparrow\rangle, \quad \sigma_-|\uparrow\rangle = |\downarrow\rangle. \tag{A.53}$$

The projector involving Γ_0 leads to the decomposition of the Killing spinor:

$$\eta = e_1|\uparrow\uparrow\uparrow\rangle + e_2|\downarrow\downarrow\uparrow\rangle + e_3|\downarrow\uparrow\downarrow\rangle + e_4|\uparrow\downarrow\downarrow\rangle. \tag{A.54}$$

Substituting this into (A.51), we arrive at the relations

$$e_1e_4 = e_2e_3 = 0, \quad -|e_1^2| - |e_2^2| + |e_3^2| + |e_4^2| = \sum|e_i^2| = -|e_1^2| - |e_3^2| + |e_2^2| + |e_4^2|. \tag{A.55}$$

Combining these relations, we conclude that $e_1 = e_2 = e_3 = 0$. In other words, the spinor has only one independent component $|\uparrow\downarrow\downarrow\rangle$ and thus it satisfies the projections

$$\Gamma_A\eta = 0. \tag{A.56}$$

This relation implies that the two-form (A.48) has only holomorphic components:[44]

$$\Omega_{ab} = -e^{3A/4}e_a^{\mathbf{A}}e_b^{\mathbf{B}}\epsilon_{AB} = -e^{3A/4}\epsilon_{ab}, \quad \Omega_{\bar{a}b} = \Omega_{\bar{a}\bar{b}} = \Omega_{ym} = 0. \tag{A.57}$$

[44]We use normalization $\epsilon_{\mathbf{12}} = \epsilon_{\overline{\mathbf{12}}} = 1$. In our conventions the determinant of the metric has the following expression in terms of *curved* epsilon tensor: $\sqrt{g} = \frac{1}{4}\epsilon_{12}\epsilon_{\overline{12}}$.

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Using this information as well as projector (A.23), we can simplify the antisymmetric part of (A.49):

$$\begin{aligned}\nabla_{[\mu}(\eta^\dagger\gamma_{\nu\lambda]}\Gamma_{\overline{12}}\eta) - \frac{i}{8}e^{-3A/2}F_{\rho\sigma[\mu}\eta^\dagger\left(\gamma_{\nu\lambda]}\Gamma_{\overline{12}}\gamma^{\rho\sigma} + \gamma^{\rho\sigma}\gamma_{\nu\lambda]}\Gamma_{\overline{12}}\right)\eta \\ + \frac{18}{48}\eta^\dagger\left(\gamma_{[\nu\lambda}\Gamma_{\overline{12}}\gamma_{\mu]}\partial\!\!\!/ A + \partial\!\!\!/ A\gamma_{\mu\nu\lambda}\Gamma_{\overline{12}}\right)\eta = 0 \\ \partial_{[\mu}\Omega_{\nu\lambda]} - \frac{i}{8}e^{-3A/2}F_{\rho\sigma[\mu}\,\eta^\dagger\gamma_{\nu\lambda]}[\Gamma_{\overline{12}},\gamma^{\rho\sigma}]\eta + \frac{3}{8}\eta^\dagger\left(\{\Gamma_{\overline{12}},\gamma_{[\nu\lambda}\}\partial_{\mu]}A + 3\partial_{[\mu}A\gamma_{\nu\lambda]}\Gamma_{\overline{12}}\right)\eta = 0\end{aligned}$$

We are interested in the situation where two of the indices (μ,ν,λ) are holomorphic, then the last equation simplifies:

$$\begin{aligned}\frac{1}{3}\partial_\mu\Omega_{ab} + \frac{1}{2}\partial_\mu A\;\Omega_{ab} &= \frac{i}{4}e^{-3A/2}\epsilon^{\overline{AB}}F_{\bar{\mathbf{B}}\sigma[\mu}\;\eta^\dagger\gamma_{ab]}[\Gamma_{\overline{A}},\gamma^\sigma]\eta \\ &= \frac{i}{4}e^{-3A/2}\Big[2F_{\bar{\mathbf{B}}\sigma[\mu}\Omega_{ab]}e^{\sigma\bar{\mathbf{B}}} - 2\epsilon^{\overline{AB}}F_{\bar{\mathbf{B}}\bar{\mathbf{A}}[\mu}\eta^\dagger\gamma_{ab]}\eta \\ &\qquad +2\epsilon^{\overline{AB}}F_{\bar{\mathbf{B}}y[\mu}\;\eta^\dagger\gamma_{ab]}\Gamma_{\overline{A}}\gamma^y\eta\Big] \\ &= \frac{i}{2}e^{-3A/2}\left[F_{\bar{\mathbf{B}}}{}^{\bar{\mathbf{B}}}{}_{[\mu}\Omega_{ab]} + 2F_{\bar{\mathbf{1}}\bar{\mathbf{2}}[\mu}\eta^\dagger\gamma_{ab]}\eta - \frac{1}{3}\delta^y_\mu\epsilon^{\overline{AB}}F_{\bar{\mathbf{B}}y[a}\;\eta^\dagger\gamma_{b]}\Gamma_{\overline{A}}\eta\right]\end{aligned} \tag{A.58}$$

Here index μ takes values $\bar{a}$ and y. It is convenient to consider these two cases separately. We begin with y-component:

$$\begin{aligned}\frac{1}{3}\partial_y\Omega_{ab} + \frac{1}{2}\partial_y A\;\Omega_{ab} &= \frac{i}{6}e^{-3A/2}\left[F_{\bar{\mathbf{B}}}{}^{\bar{\mathbf{B}}}{}_y\Omega_{ab} - \frac{1}{2}e^{3A/4}\epsilon_{ab}\epsilon^{cd}\epsilon^{\overline{AB}}F_{\bar{\mathbf{B}}yc}e_d^{\mathbf{A}}\delta_{A\bar{A}}\right] \\ &= \frac{i}{6}e^{-3A/2}\left[F_{\bar{\mathbf{B}}}{}^{\bar{\mathbf{B}}}{}_y + \frac{1}{2}F_{\bar{\mathbf{B}}yc}(-e^{c\bar{\mathbf{B}}})\right]\Omega_{ab} = \frac{i}{4}e^{-3A/2}g^{\bar{c}e}F_{\bar{c}ey}\Omega_{ab} \\ &= -\frac{e^{-3A/2}}{4}g^{\bar{c}e}\partial_y(e^{3A/4}J_{\bar{c}e})\Omega_{ab} = \left[\frac{3}{4}\partial_y A + \frac{1}{4}\partial_y\log\sqrt{\det\, g}\right]\Omega_{ab}\end{aligned}$$

At the last stage we used the relation (A.45): $J_{a\bar{b}} = e^{3A/4}g_{a\bar{b}}$.

Simplifying the last equation and using the expression for the holomorphic form (A.57), we arrive at a relation

$$\frac{1}{3}\partial_y\epsilon_{ab} = \frac{1}{4}\epsilon_{ab}\partial_y\log\sqrt{\det\, g} \tag{A.59}$$

To determine the y-dependence of $\det\, g$, we multiply the last equation by ϵ^{ab}:

$$\frac{1}{3}\epsilon^{ab}\partial_y\epsilon_{ab} = \frac{1}{2}\partial_y\log\sqrt{\det\, g},$$

and add this relation to its conjugate:

$$\frac{2}{3}\partial_y\log\sqrt{\det\, g} = \partial_y\log\sqrt{\det\, g}.$$

This leads to the conclusion that $\partial_y(\det\, g) = 0$ and to simplification in equation (A.59):

$$\partial_y\epsilon_{ab} = 0. \tag{A.60}$$

Let us now demonstrate that the anti-holomorphic derivatives $\partial_{\bar{c}}\epsilon_{ab}$ vanish as well. To do so, we go back to equation (A.58):

$$\begin{aligned}
\frac{1}{3}\partial_{\bar{c}}\Omega_{ab}+\frac{1}{2}\partial_{\bar{c}}A\ \Omega_{ab} &= \frac{i}{6}e^{-3A/2}\left[F_{\bar{\mathbf{B}}}{}^{\bar{\mathbf{B}}}{}_{\bar{c}}\ \Omega_{ab}+4F_{\bar{\mathbf{1}}\bar{\mathbf{2}}[a}\eta^\dagger\gamma_{b]\bar{c}}\eta\right]\\
&= \frac{ie^{-3A/2}}{12}\epsilon^{\overline{ef}}F_{\overline{ef}d}\left[-\epsilon_{\overline{hc}}g^{\bar{h}d}\Omega_{ab}+2\epsilon_{ab}\epsilon^{de}J_{e\bar{c}}\right]\\
&= \frac{ie^{-3A/2}}{12}\epsilon^{\overline{ef}}F_{\overline{ef}d}\left[4g_{\bar{c}h}\epsilon^{hd}-2\epsilon^{de}g_{e\bar{c}}\right]\Omega_{ab}\\
&= 3\epsilon^{\overline{ef}}\partial_{\bar{e}}A\ g_{\bar{f}d}\ g_{\bar{c}h}\ \epsilon^{hd}\ \Omega_{ab}=\frac{3}{4}\partial_{\bar{c}}A\ \Omega_{ab}.
\end{aligned}\tag{A.61}$$

At the last stage we used the relation $\epsilon_{ab}g^{b\bar{c}}=4g_{a\bar{b}}\epsilon^{\overline{bc}}$ as well as an expression for the field strength:

$$\epsilon^{\overline{ef}}F_{\overline{ef}d}=6ie^{3A/4}\epsilon^{\overline{ef}}\partial_{\bar{e}}AJ_{\bar{f}d}=-6ie^{3A/2}\epsilon^{\overline{ef}}\partial_{\bar{e}}Ag_{\bar{f}d},$$

which can be easily extracted from (A.35).

Combining equations (A.60) and (A.61), we conclude that

$$\partial_y\epsilon_{ab}=\partial_{\bar{c}}\epsilon_{ab}=0: \qquad d\left[e^{-3A/4}\Omega\right]=0. \tag{A.62}$$

Since ϵ_{ab} is function of z_2 and z_2, we can make a holomorphic change of coordinates to set

$$\epsilon_{12}=1, \quad \sqrt{g}=\frac{1}{4}\epsilon_{12}\epsilon_{\overline{12}}=\frac{1}{4}. \tag{A.63}$$

Recalling the expression (A.47) for the metric, we arrive for the relation between e^A and Kahler potential:

$$\partial_1\bar{\partial}_1K\ \partial_2\bar{\partial}_2K-\partial_2\bar{\partial}_1K\ \partial_2\bar{\partial}_1K=\frac{1}{4}e^{-3A} \tag{A.64}$$

Notice that this relation holds only in a particular coordinate frame defined by (A.63).

This completes our discussion of the equations for Killing spinors, let us summarize the results. We began with an assumption that eleven dimensional geometry was static and had SO(6) symmetry. Since we were interested in the geometries produced by membranes, we also assumed that the flux was electric. Then, solving the equations for the spinors, we arrived at the geometry:

$$ds^2=-e^{2A}dt^2+e^{A/2}\left[e^{-3A/2}dy^2+2g_{a\bar{b}}dz^ad\bar{z}^b\right]+y^2e^{-A}d\Omega_5^2 \tag{A.65}$$

$$G_4=idt\wedge d(e^{3A/2}g_{a\bar{b}}dz^a\wedge d\bar{z}^b) \tag{A.66}$$

Moreover, we demonstrated that the metric h has a simple expression in terms of a Kahler potential $K(z,\bar{z},y)$:

$$g_{a\bar{b}}=e^{3A/2}\partial_a\bar{\partial}_bK. \tag{A.67}$$

and the warp factor e^A is determined by (A.64). Thus the solution is uniquely parameterized by one real function K and in the next subsection we will use the equations fluxes to find the restrictions on this function.

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A.3 Equations for the flux

Looking at the solution (A.65), we can easily write down the equation of motion for the flux G_4:

$$d[e^{-7A/2-A/4}y^5 *_5 d(e^{3A/4}J)] = 0 \tag{A.68}$$

Here five-dimensional Hodge duality is taken with respect to the metric which appears in the square brackets in (A.65) and the two-form J has been introduced before (see (A.45)):

$$J = e^{3A/4}g_{a\bar{b}}dz^a \wedge d\bar{z}^b \tag{A.69}$$

We begin the study of (A.68) with analysis of the terms which do not contain dy:

$$d[e^{-7A/2-A/4+3A/4}y^5 *_4 \partial_y(e^{3A/2}\hat{J})] = 0 = d_4\left[e^{-3A} *_4 \partial_y(e^{3A/4}J)\right] \tag{A.70}$$

Using the relation (A.37) and anti-self-duality of J, we can rewrite the last equation:

$$0 = d_4\left[e^{-9A/4}\left(-\partial_y(*J) + \frac{9}{4}\partial_y A * J\right)\right] = d_4\partial_y(e^{-9A/4}J)$$

This is an integrability condition for the equation which has been encountered before.

Next we look at the y-component of the field equation (A.68):

$$\partial_y[e^{-3A}y^5 *_4 \partial_y(e^{3A/4}J)] - d_4[e^{-9A/2}y^5 *_4 d_4(e^{3A/4}J)] = 0 \tag{A.71}$$

To rewrite the first term we again use equation (A.37) and anti-self-duality of J:

$$e^{-3A} *_4 \partial_y(e^{3A/4}J) = \partial_y(e^{-9A/4}J) = \partial_y\partial_a\bar{\partial}_b K dz^a \wedge d\bar{z}^b, \tag{A.72}$$

and to simplify the second term, we use equation (A.36) as well as anti-self-duality of J:

$$\begin{aligned} d_4[e^{-9A/2} *_4 d_4(e^{3A/4}J)] &= 3d_4[e^{-15A/4} *_4 (d_4A \wedge J)] = -3d_4[e^{-15A/4}J_m{}^p\partial_p A dx^m] \\ &= -3d_4[e^{-3A}(\partial_a A dz^a - \bar{\partial}_a A d\bar{z}^a)] = -2\partial_a\bar{\partial}_b e^{-3A} dz^a \wedge d\bar{z}^b \end{aligned}$$

To go to the second line we used the relations

$$J_a{}^b = e^{3A/4}\delta_a^b, \quad J_{\bar{a}}{}^{\bar{b}} = -e^{3A/4}\delta_{\bar{a}}^{\bar{b}}. \tag{A.73}$$

Using all this information, equation (A.71) can be rewritten as

$$\Delta_y\partial_a\bar{\partial}_b K + 2\partial_a\bar{\partial}_b e^{-3A} = 0 \tag{A.74}$$

Since (anti)holomorphic functions can be added to the Kahler potential, we can choose the gauge where

$$\Delta_y K + 2e^{-3A} = 0 \tag{A.75}$$

A.4 Solution with $\mathrm{SO}(6)\times\mathrm{U}(1)^2\times\mathrm{U}(1)_t$ isometry

In this appendix we constructed the most general $\mathrm{SO}(6)\times\mathrm{U}(1)_t$-invariant geometry which is produced by supersymmetric membranes. However, in order to describe the string webs in IIB string theory, we are interested in the solutions with two extra isometries (see ansatz (A.2)). In this subsection we will discuss the additional restrictions which are imposed by these symmetries. First of all, it is clear that the coordinates w_1 and w_2 are orthogonal to the y-direction, so one can repeat the earlier arguments to arrive at the geometry (A.15), but now the four-dimensional metric is given by

$$g_{ij}dx^i dx^j = \hat{g}_{\alpha\beta}dw^\alpha dw^\beta + \tilde{g}_{MN}dr^M dr^N \tag{A.76}$$

and all functions are invariant under translations in w_1, w_2. Our goal is to establish the connection between w_α and the complex coordinates z_a. To simplify the w-components of the equation (A.17) we observe that the ansatz (A.2) implies that

$$\not{F} = 3\gamma^\alpha \not{W}_\alpha. \tag{A.77}$$

Since we are planning to perform a reduction and T duality in the isometry directions, the Killing spinor should not depend on w_α, then the differential equation (A.17) simplifies:

$$\frac{1}{4}\omega_\alpha\eta - \frac{ie^{-3A/2}}{16}\gamma^\beta \not{W}_\beta\gamma_\alpha\eta = 0 \tag{A.78}$$

To extract an expression for $W_\alpha\eta$, we compare the last relation with equation (A.23):

$$\not{\partial}A\eta - \frac{ie^{-3A/2}}{6}\gamma^\beta \not{W}_\beta\eta = 0. \tag{A.79}$$

Since γ_α commutes with $\not{W}$, the last two equations can be combined to yield

$$\omega_\alpha\eta + \frac{3}{2}\gamma_\alpha\not{\partial}A\eta - \frac{ie^{-3A/2}}{2}\not{W}_\alpha\eta = 0 \tag{A.80}$$

Applying an operator $\gamma_y(1-i\Gamma_y\Gamma_S)$ to this relation and using projector (A.18), we arrive at the relation which does not contain Γ_y:[45]

$$-\partial_y g_{\alpha\beta}\gamma^\beta\eta - \frac{3}{2}\gamma_\alpha\partial_y A\eta - ie^{-3A/2}(W_\alpha)_{yM}\gamma^M\eta = 0 \tag{A.81}$$

Assuming that a tensor $(W_\alpha)_{yM}$ has at least one nontrivial component,[46] we find a projector involving γ_α and γ_M:

$$(a^\alpha\gamma_\alpha + b^M\gamma_M)\eta = 0 \tag{A.82}$$

[45] To arrive at this equation we used an expression for the spin connection: $\omega_\alpha = -(\not{\partial}g_{\alpha\beta})\gamma^\beta$.

[46] If all components of $(W_\alpha)_{yM}$ vanish, then, by applying an operator $(1-i\Gamma_y\Gamma_S)$ to the relation (A.80), we can again arrive at (A.82) as long as $\not{F}$ is not identically equal to zero. The latter case is degenerate and we will not discuss it further.

Due to projectors (A.18) and (A.20), the product of four gamma matrices acts on the Killing spinor in a very simple way, so we can find another relation which is "dual" to (A.82). These two relations can be combined to give independent projectors:

$$\gamma_M \eta = i a_M^\alpha \gamma_\alpha \eta = i(a_M^\alpha e_\alpha^{\mathbf{a}})\hat{\Gamma}_{\mathbf{a}}\eta. \tag{A.83}$$

Notice that $E_M^{\mathbf{a}} = a_M^\alpha e_\alpha^{\mathbf{a}}$ can be viewed as a veilbein in the r_1, r_2 directions:

$$2\tilde{g}_{MN}\eta = \{\gamma_M, \gamma_N\}\eta = E_M^{\mathbf{a}} E_N^{\mathbf{b}} \{\hat{\Gamma}_{\mathbf{a}}, \hat{\Gamma}_{\mathbf{b}}\}\eta = 2\delta_{\mathbf{ab}} E_M^{\mathbf{a}} E_N^{\mathbf{b}}\eta. \tag{A.84}$$

We can also use reparameterizations in r_1, r_2 subspace to set $a_M^\alpha = \delta_M^\alpha$. Assuming that the coordinates and the vielbein are chosen in this fashion, equation (A.83) can be rewritten as

$$(\tilde{\gamma}_M - i\delta_M^\alpha \hat{\gamma}_\alpha)\eta = 0. \tag{A.85}$$

These relations must be consistent with holomorphic projectors (A.56), so, going to complex coordinates and using (A.56), we can rewrite the last relation as

$$\left(\frac{\partial \bar{z}^a}{\partial r^M} - i\delta_M^\alpha \frac{\partial \bar{z}^a}{\partial w^\alpha}\right)\gamma_{\bar{a}}\eta = 0. \tag{A.86}$$

Since spinor η is constrained by (A.56), the last relation can be satisfied only if the coefficients in front of $\gamma_{\bar{1}}$ and $\gamma_{\bar{2}}$ vanish separately, so we find

$$\frac{\partial \bar{z}^a}{\partial r^m} - i\frac{\partial \bar{z}^a}{\partial w^m} = 0: \qquad \bar{z}^a = \bar{z}^a(r_1 - iw_1, r_2 - iw_2). \tag{A.87}$$

By making a holomorphic reparameterization, we can choose convenient coordinates coordinates:

$$z_1 = r_1 + iw_1, \qquad z_2 = r_2 + iw_2. \tag{A.88}$$

Notice that apriori the new coordinates are not consistent with the gauge choice which led to (A.75), so we have to go back to a more general equation (A.74). In the present case the Kahler potential does not depend on w, so both holomorphic and anti-holomorphic derivatives should be replaced by the variations with respect to corresponding r and equation (A.74) becomes

$$\partial_M \partial_N(\Delta_y K + 2e^{-3A}) = 0: \qquad \Delta_y K + 2e^{-3A} = h(y) \tag{A.89}$$

Since Kahler potential can be shifted by an arbitrary function of y without affecting the metric, we can again impose the gauge (A.75):

$$\Delta_y K + 2e^{-3A} = 0. \tag{A.90}$$

To summarize, we showed that solutions which have two translational isometries, in addition to $\mathrm{SO}(6)\times\mathrm{U}(1)_t$, are still described by the system (A.65), (A.64), (A.75), and the isometry directions must be related with complex coordinates in a very simple way (A.88).

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A.5 Summary of the solution

Let us collect the results derived in this appendix. We have found the most general eleven-dimensional geometry which preserved eight supercharges along with $\mathrm{SO}(6)\times\mathrm{U}(1)_t$ bosonic isometries:

$$\begin{aligned} ds^2 &= -e^{2A}dt^2 + 2e^{2A}h_{a\bar{b}}dz^a d\bar{z}^b + e^{-A}(dy^2 + y^2 d\Omega_5^2) \\ G_4 &= idt\wedge d(e^{3A}h_{a\bar{b}}dz^a\wedge d\bar{z}^b), \quad h_{a\bar{b}} = \partial_a\bar{\partial}_b K. \end{aligned} \tag{A.91}$$

The solution is parameterized in terms of the Kahler potential $K(z,\bar{z},y)$ which should satisfy two differential equations (A.64), (A.75):

$$\partial_1\bar{\partial}_1 K\ \partial_2\bar{\partial}_2 K - \partial_2\bar{\partial}_1 K\ \partial_2\bar{\partial}_1 K = -\frac{1}{8}\Delta_y K, \quad \Delta_y K = -2e^{-3A}. \tag{A.92}$$

It is easy to guess a generalization of the solution (A.91) to the situations without SO(6) isometries:

$$ds^2 = -e^{2A}dt^2 + 2e^{2A}h_{a\bar{b}}dz^a d\bar{z}^b + e^{-A}d\mathbf{y}_6 \tag{A.93}$$

$$G_4 = idt\wedge d(e^{3A}h_{a\bar{b}}dz^a\wedge d\bar{z}^b), \quad h_{a\bar{b}} = \partial_a\bar{\partial}_b K. \tag{A.94}$$

Starting with this ansatz, we can explicitly check that conditions for supersymmetry and equations of motion are satisfied as long as Kahler potential obeys the differential equations (A.64), (A.75).[47]

To find the geometries produced by the string webs, one needs to look at eleven dimensional geometries which have two extra translational isometries in addition to $\mathrm{U}(1)_t\times\mathrm{SO}(6)$. The restrictions coming from this requirement were discussed in the last subsection where we found that the complex coordinates must have the form

$$z_1 = r_1 + iw_1, \qquad z_2 = r_2 + iw_2, \tag{A.95}$$

where w_1 and w_2 are the isometry directions. To find the IIB solution describing the string web, we write the eleven dimensional metric in a slightly more explicit form:

$$ds_M^2 = -e^{2A}dt^2 + e^{2A}h_{ab}(dw^a dw^b + dr^a dr^b) + e^{-A}d\mathbf{y}_6 \tag{A.96}$$

$$G_4 = dt\wedge d(e^{3A}h_{ab}\ dr^a\wedge dw^b), \quad h_{ab} = \frac{1}{2}\partial_a\partial_b K \tag{A.97}$$

Reducing this geometry along w_1 and T dualizing along w_2, we find the solution in IIB SUGRA:

$$\begin{aligned} ds^2_{\mathrm{IIB}} &= \sqrt{h_{11}}\left[-e^{3A}dt^2 + e^{3A}h_{ab}dr^a dr^b + \frac{e^{-3A}}{\det h}dw_2^2 + d\mathbf{y}_6^2\right] \\ e^{2\Phi} &= \frac{h_{11}^2}{\det h}, \quad C^{(0)} = -\frac{h_{12}}{h_{11}}, \quad B = e^{3A}h_{1a}\ dt\wedge dr^a, \quad C^{(2)} = e^{3A}h_{2a}\ dt\wedge dr^a \end{aligned} \tag{A.98}$$

Recalling that equation (A.64) implies a very simple expression for the determinant ($\det h = e^{-3A}$), we arrive at the metric in the Einstein frame:

$$ds_E^2 = e^{-3A/4}\left[-e^{3A}dt^2 + e^{3A}h_{ab}dr^a dr^b + dw_2^2 + d\mathbf{y}_6^2\right]. \tag{A.99}$$

[47] Similar checks for other brane intersections were performed in [17].

B. Embedding of 1/2-BPS bubbling geometries

In section 6 we discussed bubbling solutions preserving eight supercharges, in particular, 1/2-BPS geometries constructed in [20] fall into this category. However, embedding of 1/2-BPS geometries into the general ansatz (6.2) requires some algebraic manipulations, and we present them in this appendix. We will also embed the 1/2-BPS solutions of M theory [20] into the ansatz (7.5).

B.1 IIB supergravity

Ten-dimensional case was discussed in section 6.2, where we wrote down an embedding of 1/2-BPS solutions of [20] into the general 1/4-BPS geometry (6.2). Here we will derive the relations (6.26)–(6.29).

Half-BPS geometries constructed in [20] were parameterized by one harmonic function $\tilde{Z}(z,\bar{z};x)$, and, with slight change of notation, the metric can be written as

$$\begin{aligned} ds^2 &= -\tilde{h}^{-2}(dt+V)^2+\tilde{h}^2\ (dx^2\ +dzd\bar{z})+xe^{H}d\Omega_3^2+xe^{-H}d\tilde{\Omega}_3^2, \\ \tilde{h}^{-2} &= 2x\cosh H, \quad (dV)_{z\bar{z}} = \frac{i}{2x}\partial_x\tilde{Z}, \quad x\partial_x V=i(dz\partial_z-d\bar{z}\partial_{\bar{z}})\tilde{Z}, \quad \tilde{Z}=\frac{1}{2}\tanh H, \\ 4\partial_z\partial_{\bar{z}}\tilde{Z}&+x\partial_x\left(\frac{\partial_x\tilde{Z}}{x}\right) = 0. \end{aligned} \tag{B.1}$$

To compare this with (6.2), we identify the three dimensional sphere appearing in (6.2) with S^3 in the metric (B.1), while embedding the Killing direction ψ from (6.2) into $\tilde{S}^3$:

$$d\tilde{\Omega}_3^2=d\theta^2+\cos^2\theta d\psi^2+\sin^2\theta d\tilde{\phi}^2. \tag{B.2}$$

As discussed in section 6.2, this identification follows naturally from the analysis of the R symmetry group. Once the embedding of S^3 and ψ is specified, we can compare the appropriate warp factors in (6.2) and (B.1), this leads to the following relations:

$$y=x\cos\theta, \quad e^G=\frac{e^H}{\cos\theta}, \quad Z=\frac{1}{2}\tanh G=\frac{1}{2}\frac{e^H-e^{-H}c_\theta^2}{e^H+e^{-H}c_\theta^2}. \tag{B.3}$$

By comparing the coordinate dependence of the Killing spinor in 1/2-and 1/4-BPS cases, one concludes that an appropriate coordinate on the Kahler base in (6.2) is $\phi=\tilde{\phi}+t$ rather than $\tilde{\phi}$. Implementing this shift in (B.1)–(B.2) and comparing with (6.2), we reproduce the correct g_{tt} and find the expression for one-form ω:

$$\omega=\frac{\cosh H}{\cos\theta\cosh G}V+\frac{e^{-G}\tan^2\theta}{e^G+e^{-G}}d\phi. \tag{B.4}$$

Let us now extract the metric of the four-dimensional Kahler space appearing in (6.2). We begin with looking at the line element in the (x,θ) subspace of (B.1) and subtracting the dy^2 term from (6.2):

$$\begin{aligned} ds^2_{x,\theta,\perp} &= \frac{dx^2}{2x\cosh H}+xe^{-H}d\theta^2-\frac{dy^2}{2y\cosh G}=\frac{y\cosh H}{e^H\cosh G}\left[\frac{e^H\sin\theta dx}{2y\cosh H}+d\theta\right]^2 \\ &= \frac{e^{-H}\cosh H}{y\cosh G}\left[d(x\sin\theta)+\left(\tilde{Z}-\frac{1}{2}\right)\sin\theta dx\right]^2 \end{aligned} \tag{B.5}$$

To proceed its is convenient to introduce a new function[48] D:

$$\tilde{Z} - \frac{1}{2} \equiv -x\partial_x D. \tag{B.6}$$

Rewriting the metric (B.5) in terms of D, one finds

$$ds^2_{x,\theta,\perp} = \frac{e^{-H}\cosh H}{y\cosh G}\left[e^D d(x\sin\theta e^{-D}) + x\sin\theta d_2 D\right]^2 \tag{B.7}$$

Here d_2 denotes a differential along the directions $z, \bar{z}$. The structure of equation (B.7) suggests that it is convenient to trade variables (x,θ) for new coordinates

$$y = x\cos\theta, \qquad W = x\sin\theta e^{-D}. \tag{B.8}$$

In particular, as we have shown, these two coordinates are orthogonal to each other.

Notice that the relation (B.6) does not determine D uniquely: any function of $(z,\bar{z})$ can be added to it without affecting (B.6). To fix this ambiguity, we rewrite the expression for x-derivative of V in terms of D (see (B.1):

$$x\partial_x V = i(dz\partial_z - d\bar{z}\partial_{\bar{z}})\left[\tilde{Z} - \frac{1}{2}\right] = -ix\partial_x(dz\partial_z - d\bar{z}\partial_{\bar{z}})D. \tag{B.9}$$

This relation implies that D can be determined completely, by requiring that

$$V = -i(dz\partial_z - d\bar{z}\partial_{\bar{z}})D \tag{B.10}$$

in addition to (B.10).

Let us now look at the t–ϕ subspace. We already extracted the expression for ω, and now we evaluate the rest of the metric:

$$\begin{aligned} ds^2_{t,\phi} &\equiv -2x\cosh H(dt+V)^2 + xe^{-H}\sin^2\theta(d\phi - dt)^2 \\ &= -h^{-2}(dt+\omega)^2 + \frac{x\cosh H}{e^G\cosh G}\tan^2\theta(V+d\phi)^2 \\ &\quad + \left[y^2h^2e^{-2G}\tan^4\theta - \frac{x\cosh H}{e^G\cosh G}\tan^2\theta + ye^{-G}\tan^2\theta\right]d\phi^2 \end{aligned} \tag{B.11}$$

Simplifications show that the last line gives vanishing contribution:

$$\frac{e^{-G}}{2\cosh G}\tan^2\theta - \frac{\cosh H}{\cosh G}\frac{1}{\cos\theta} + 1 = \frac{e^{-G}}{2\cosh G}\tan^2\theta - \frac{e^{-H}}{2\cosh G}\frac{\sin^2\theta}{\cos\theta} = 0.$$

Using expressions (B.7) and (B.11), we can extract the Kahler metric appearing in (6.2):

$$\begin{aligned} 2\partial_a\bar{\partial}_b K dz^a d\bar{z}^b &= \frac{Z+\frac{1}{2}}{h^2}\left[\frac{e^{-H}\cosh H}{y\cosh G}\left\{\frac{y^2}{\cot^2\theta}(V+d\phi)^2 + e^{2D}(dW + Wd_2D)^2\right\} + \frac{dzd\bar{z}}{\tilde{h}^{-2}}\right] \\ &= \left[\frac{\cosh H}{\cosh G}\frac{e^{2D}}{\cos\theta}\left\{W^2(V+d\phi)^2 + (dW+Wd_2D)^2\right\} + ye^G\tilde{h}^2dzd\bar{z}\right] \end{aligned} \tag{B.12}$$

[48]This definition was inspired by the discussion of the M theory case, where function D arises in a more natural way (see next subsection).

Let us try to guess the complex structure. We already have a natural complex structure in two dimensional space spanned by z, $\bar{z}$, so only the terms inside curly bracket in (B.12) need additional analysis. The metric appearing there can be simplified using the expression (B.10) for V:

$$\begin{aligned} ds_2^2 &= W^2\left(d\phi - i\{\partial_z D dz - \partial_{\bar z} D d\bar z\}\right)^2 + \left(dW + W\{\partial_z D dz + \partial_{\bar z} D d\bar z\}\right)^2 \\ &= dW^2 + W^2 d\phi^2 + 4W^2 \partial_z D \partial_{\bar z} D dz d\bar z + \left[2\partial_z D dz (W dW - iW^2 d\phi) + c.c.\right]. \end{aligned}$$

This relation suggests a natural complex coordinate $w = We^{i\phi}$:

$$ds_2^2 = (dw + 2w\partial_z D dz)(d\bar w + 2\bar w \partial_{\bar z} D d\bar z). \tag{B.13}$$

Now the Kahler metric and one-form ω can be rewritten in terms of complex coordinates z and w:

$$\begin{aligned} 2\partial_a \bar\partial_b K dz^a d\bar z^b &= \left[\frac{\cosh H}{\cosh G}\frac{e^{2D}}{\cos\theta}\left|dw + 2w\tilde\partial_z D dz\right|^2 + y e^G \tilde h^2 dz d\bar z\right] \\ \omega &= -i\frac{\cosh H}{\cos\theta\cosh G}(\tilde\partial_z D dz - \tilde\partial_{\bar z} D d\bar z) - \frac{i}{2}\frac{e^{-G}\tan^2\theta}{e^G + e^{-G}} d\log\frac{w}{\bar w} \end{aligned} \tag{B.14}$$

Notice that the derivatives $\tilde\partial_z$ and $\tilde\partial_{\bar z}$ are taken at constant θ and x, in contrast to ∂_z and $\partial_{\bar z}$ which are taken at constant y and W.

To find the relation between D and Kahler potential parameterizing 1/4-BPS geometry (6.2), we evaluate the determinant of the metric (B.14):

$$\det h_{a\bar b} = \frac{1}{4}\frac{e^{G+2D}}{e^G + e^{-G}} = \frac{1}{4}\left(Z + \frac{1}{2}\right)e^{2D} \tag{B.15}$$

Comparing this with corresponding equation in (6.2) (and choosing the gauge $W(z) = \frac{1}{2}$ there), we relate D and derivative of the Kahler potential:

$$y^{-1}\partial_y K = 2D - \log y. \tag{B.16}$$

To summarize, we showed that 1/2-BPS bubbling solutions constructed in [20] can be embedded in the more general 1/4-BPS ansatz. To construct the appropriate map, one should first rewrite the 1/2-BPS geometries (B.1) in terms of function D rather than $\tilde Z$:

$$x\partial_x D = \frac{1}{2} - \tilde Z, \qquad V = -i(dz\partial_z - d\bar z \partial_{\bar z})D. \tag{B.17}$$

Then, defining a new variable y and holomorphic coordinates z, w:

$$y = x\cos\theta, \quad w = x\sin\theta e^{-D+i\phi} \tag{B.18}$$

one arrives at the map (B.16) between (B.1)–(B.2) and (6.2).

In section 6.2 we also needed the expression for ∂_x (which is taken at constant z, θ) in terms of ∂_y (which is computed for fixed z, w). To find the desired relation, we consider various differentials at constant values of $(z, \bar z, w, \bar w)$:

$$dW = e^{-D}\left[s_\theta(1 - x\partial_x D)dx + c_\theta x d\theta\right] = 0: \ dy = c_\theta dx - x s_\theta d\theta = \frac{dx}{c_\theta}(1 - x s_\theta^2 \partial_x D)$$

This leads to a general expression for ∂_y in terms of ∂_x, and we will be particularly interested in its implications for the derivatives of D:

$$\partial_y = \frac{c_\theta}{1 - s_\theta^2 x \partial_x D}\partial_x, \quad \partial_y D = \frac{c_\theta \partial_x D}{1 - s_\theta^2 x \partial_x D}. \tag{B.19}$$

We conclude this subsection by noticing that the linear equation obeyed by $\tilde{Z}$ implies a simple Laplace equation for function D:

$$4\partial_z \partial_{\bar{z}} D + x^{-1}\partial_x(x\partial_x D) = 0. \tag{B.20}$$

B.2 M theory

Let us now embed the 1/2-BPS eleven dimensional solutions constructed in [20] into the general 1/4-BPS ansatz (7.5). We begin with recalling the metric found in [20]:

$$\begin{aligned} ds^2 &= -4e^{2\lambda}\cosh^2\xi(d\tau + V)^2 + \frac{e^{-4\lambda}}{\cosh^2\xi}\left[dx^2 + e^{\tilde{D}}dzd\bar{z}\right] + 4e^{2\lambda}d\Omega_5^2 + x^2 e^{-4\lambda}d\Omega_2^2 \\ e^{-6\lambda} &= \frac{\partial_x \tilde{D}}{x(1 - x\partial_x \tilde{D})}, \quad V = \frac{i}{2}(\partial_z - \partial_{\bar{z}})\tilde{D}, \quad \sinh\xi = xe^{-3\lambda} \end{aligned} \tag{B.21}$$

Comparing this with (7.5):

$$ds^2 = -\frac{4e^{2\lambda}}{9}\cosh^2\zeta(dt + \rho)^2 + 4e^{2\lambda}d\Omega_5^2 + e^{-4\lambda}\left(h_{ij}dx^i dx^j + \frac{dy^2}{\cosh^2\zeta}\right), \tag{B.22}$$

we observe that function e^λ appearing in both metrics is the same. To make further comparison, we write the metric on S^2 as

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\tilde{\phi}^2 \tag{B.23}$$

and introduce a shift and a rescaling:[49] $\tilde{\phi} = \phi - 2\tau, \quad \tau = \frac{t}{3}$. This leads to identifications:

$$\begin{aligned} \cosh^2\zeta &= \cosh^2\xi - \sinh^2\xi\sin^2\theta = 1 + (\sinh\xi\cos\theta)^2 : \quad y = x\cos\theta. \\ \rho &= 3V + 3\frac{x^2\sin^2\theta e^{-6\lambda}}{1 + y^2 e^{-6\lambda}}\left(V + \frac{1}{2}d\phi\right) = 3V\frac{\cosh^2\xi}{\cosh^2\zeta} + \frac{3}{2}\tan^2\theta\tanh^2\zeta d\phi \end{aligned} \tag{B.24}$$

To extract the metric of the four dimensional Kahler corresponding to (B.21), we begin with looking at the x–θ subsector:

$$\begin{aligned} ds^2_{x,\theta,\perp} &\equiv \frac{e^{-4\lambda}dx^2}{\cosh^2\xi} + x^2 e^{-4\lambda}d\theta^2 - \frac{e^{-4\lambda}dy^2}{\cosh^2\zeta} = \frac{e^{-4\lambda}}{\cosh^2\zeta}\left(\frac{s_\theta dx}{ch_\xi} + xc_\theta ch_\xi d\theta\right)^2 \\ &= e^{-4\lambda}\frac{ch_\xi^2}{ch_\zeta^2}\left(d(xs_\theta) - xs_\theta\partial_x\tilde{D}dx\right) = e^{-4\lambda}\frac{ch_\xi^2}{ch_\zeta^2}\left(e^{\tilde{D}}d(xe^{-\tilde{D}}s_\theta) + xs_\theta d_2\tilde{D}\right)^2 \end{aligned} \tag{B.25}$$

[49] This change of variables can be extracted by comparing Killing spinors for 1/2 and 1/4-BPS solutions, but we will not present the argument here.

Here d_2 denotes a differential along the directions z, $\bar{z}$. To simplify the last expression the following relation was used:

$$\frac{1}{ch^2\xi} - 1 = -\frac{x^2 e^{-6\lambda}}{1+x^2e^{-6\lambda}} = -x\partial_x\tilde{D} \tag{B.26}$$

The structure of equation (B.25) suggests that it is convenient to trade variables (x,θ) for new coordinates

$$y = x\cos\theta, \qquad W = x\sin\theta e^{-\tilde{D}}. \tag{B.27}$$

In particular, as we have shown, these two coordinates are orthogonal to each other.

Let us now look at the t–ϕ sector. We already extracted the expressions for g_{tt} and ρ, and now we evaluate the rest of the metric:

$$\begin{aligned} ds^2_{t,\phi} &\equiv -4e^{2\lambda}\cosh^2\xi\left(\frac{dt}{3}+V\right)^2 + x^2e^{-4\lambda}\sin^2\theta\left(d\phi-\frac{2}{3}dt\right)^2 \\ &= -\frac{4e^{2\lambda}}{9}\cosh^2\zeta(dt+\rho)^2 + 4e^{2\lambda}\frac{ch^2_\xi sh^2_\xi}{ch^2_\zeta}s^2_\theta\left(V+\frac{1}{2}d\phi\right)^2 \\ &\quad +x^2e^{-4\lambda}\sin^2\theta d\phi^2 - e^{2\lambda}\frac{ch^2_\xi sh^2_\xi}{ch^2_\zeta}s^2_\theta d\phi^2 + e^{2\lambda}\frac{sh^4_\xi}{ch^2_\zeta}s^4_\theta d\phi^2 \end{aligned} \tag{B.28}$$

Simplifications show that the last line gives vanishing contribution:

$$sh^2_\xi - \frac{ch^2_\xi sh^2_\xi}{ch^2_\zeta} + \frac{sh^4_\xi}{ch^2_\zeta}s^2_\theta = \frac{1}{ch^2_\zeta}(sh^2_\xi + sh^4_\xi - ch^2_\xi sh^2_\xi) = 0 \tag{B.29}$$

Using expressions (B.25) and (B.28), we can easily extract the Kahler metric appearing in (B.22):

$$h_{ij}dx^idx^j = \frac{ch^2_\xi}{ch^2_\zeta}e^{2\tilde{D}}\left[\left(dW + Wd_2\tilde{D}\right)^2 + 4w^2\left(V+\frac{1}{2}d\phi\right)^2\right] + \frac{e^{\tilde{D}}dzd\bar{z}}{\cosh^2\xi} \tag{B.30}$$

Let us try to guess the complex structure. We already have a natural complex structure in two dimensional space spanned by z, $\bar{z}$, so we only the square bracket in (B.30) needs additional analysis. The metric appearing there can be simplified using the expression (B.21) for V:

$$\begin{aligned} ds^2_2 &= (dW + W\{\partial_z Ddz + \partial_{\bar{z}}Dd\bar{z}\})^2 + W^2\left(d\phi + i\{\partial_z Ddz - \partial_{\bar{z}}Dd\bar{z}\}\right)^2 \\ &= dW^2 + W^2d\phi^2 + 4W^2\partial_z D\partial_{\bar{z}}Ddzd\bar{z} + \left[2\partial_z Ddz(WdW + iW^2d\phi) + c.c.\right]. \end{aligned}$$

This relation suggests a natural complex coordinate $w = We^{-i\phi}$:

$$ds^2_2 = (dw + 2w\partial_z Ddz)(d\bar{w} + 2\bar{w}\partial_{\bar{z}}Dd\bar{z}). \tag{B.31}$$

Now the Kahler metric and one-form ρ can be rewritten in terms of z and w:

$$\begin{aligned} h_{ij}dx^idx^j &= \frac{\cosh^2\xi e^{2\tilde{D}}}{\cosh^2\zeta}(dw + 2w\tilde{\partial}_z Ddz)(d\bar{w} + 2\bar{w}\tilde{\partial}_{\bar{z}}Dd\bar{z}) + \frac{e^{\tilde{D}}dzd\bar{z}}{\cosh^2\xi} \\ \rho &= \frac{3\cosh^2\xi}{2\cosh^2\zeta}i(\tilde{\partial}_z\tilde{D}dz - \tilde{\partial}_{\bar{z}}\tilde{D}d\bar{z}) + \frac{3i}{4}\tan^2\theta\tanh^2\zeta d\log\frac{w}{\bar{w}} \end{aligned} \tag{B.32}$$

Notice that the derivatives $\tilde{\partial}_z$ and $\tilde{\partial}_{\bar{z}}$ are taken at constant θ and x, in contrast to ∂_z and $\partial_{\bar{z}}$ which are taken at constant y and w. The relation between these two sets will be found below. Even without knowing such map, we can extract a very useful relation by taking a determinant of the metric (B.32):

$$\det h_{a\bar{b}} = \frac{e^{3\tilde{D}}}{4\cosh^2\zeta} \tag{B.33}$$

Comparing this with equation (7.6), we conclude that $D = \tilde{D}$. In principle, the relations

$$D(We^{-i\phi}, z; y) = \tilde{D}(z; x), \quad y = x\cos\theta, \quad W = x\sin\theta e^{-\tilde{D}} \tag{B.34}$$

provide a complete embedding of 1/2-BPS states [20] into the more general ansatz (7.5), but to gain a better understanding of this map we will now study it in more detail.

It is very useful to relate various derivatives appearing in the description of 1/2-BPS states with their counterparts for 1/4-BPS geometries. To find the relation between $(\tilde{\partial}_z, \tilde{\partial}_{\bar{z}})$ and $(\partial_z, \partial_{\bar{z}})$, we consider a variation of D keeping θ and x (and thus y) fixed:

$$\tilde{\partial}_i \tilde{D} dX^i = \partial_i \tilde{D}|_{yW} dX^i + \partial_W \tilde{D}|_{Xy} dW = \partial_i \tilde{D}|_{yw} dX^i - \partial_W \tilde{D}|_{Xy} W \tilde{\partial}_i \tilde{D} dX^i \tag{B.35}$$

Solving this equation we find the expression for $\tilde{\partial}_z \tilde{D}$:

$$\tilde{\partial}_z \tilde{D} = \frac{\partial_z \tilde{D}}{1 + W\partial_W \tilde{D}} = \frac{\partial_z \tilde{D}}{1 + w\partial_w \tilde{D} + \bar{w}\partial_{\bar{w}} \tilde{D}} \tag{B.36}$$

and a similar relation for $\tilde{\partial}_{\bar{z}} \tilde{D}$. To simplify this further, we need to evaluate the derivative $\partial_W \tilde{D}|_{X,y} = \partial_x \tilde{D} \cdot \partial_W x|_{X,y}$:

$$\begin{aligned}
dW|_{y,X} &= \partial_x(xe^{-\tilde{D}})s_\theta dx + xc_\theta e^{-\tilde{D}} d\theta|_{y,X} = \left[\partial_x(xe^{-\tilde{D}})s_\theta + xc_\theta e^{-\tilde{D}} \frac{c_\theta^2}{ys_\theta}\right] dx \\
&= e^{-\tilde{D}} \left[-xs_\theta \partial_x \tilde{D} + \frac{1}{s_\theta}\right] dx = e^{-\tilde{D}} \frac{\cosh^2\zeta}{s_\theta \cosh^2\xi} dx \\
\partial_W \tilde{D}|_{X,y} &= e^{\tilde{D}} \partial_x \tilde{D} \frac{s_\theta \cosh^2\xi}{\cosh^2\zeta} = \frac{e^{\tilde{D}} x s_\theta e^{-6\lambda}}{\cosh^2\zeta} = \frac{\tanh^2\zeta}{w\cot^2\theta}, \qquad \frac{1}{1 + W\partial_W \tilde{D}} = \frac{\cosh^2\zeta}{\cosh^2\xi}.
\end{aligned}$$

Here we used equation (B.26) and definitions of ζ and ξ. Now the expression for ρ (B.32) can be rewritten in terms of derivatives appropriate for the 1/4-BPS case:

$$\rho = \frac{3i}{2}(\partial_z \tilde{D} dz - \partial_{\bar{z}} \tilde{D} d\bar{z}) + \frac{3i}{4} W\partial_W \tilde{D} d\log\frac{w}{\bar{w}} = \frac{3i}{2}(dz\partial_z + dw\partial_w - cc)\tilde{D} \tag{B.37}$$

This equation agrees with (7.6).

To find the relation between ∂_x and ∂_y, we consider various differentials at constant values of $z, \bar{z}, W$:

$$dW = e^{-\tilde{D}} \left[s_\theta(1 - x\partial_x \tilde{D})dx + c_\theta x d\theta\right] = 0: \ dy = c_\theta dx - xs_\theta d\theta = \frac{dx}{c_\theta}(1 - xs_\theta^2 \partial_x \tilde{D}).$$

This leads to a general expression for ∂_y in terms of ∂_x, and we will be interested in its implications for the derivatives of $\tilde{D}$:

$$\partial_y = \frac{c_\theta}{1 - s_\theta^2 x \partial_x \tilde{D}} \partial_x, \quad \partial_y \tilde{D} = \frac{c_\theta \partial_x \tilde{D}}{1 - s_\theta^2 x \partial_x \tilde{D}} \tag{B.38}$$

Recalling the expression (B.26) for the x-derivative of $\tilde{D}$, we can simplify the last relation:

$$\partial_y \tilde{D} = \frac{c_\theta x e^{-6\lambda}}{1 + c_\theta^2 x^2 e^{-6\lambda}} = \frac{y e^{-6\lambda}}{1 + y^2 e^{-6\lambda}} \tag{B.39}$$

Comparing this with similar relation in (7.6), we conclude that

$$\partial_y (D - \tilde{D}) = 0 \tag{B.40}$$

This serves as a consistency check of the map (B.34).

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